FILTERED cA_∞-CATEGORIES AND FUNCTOR CATEGORIES

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ABSTRACT. We develop the basic theory of curved A_∞-categories (cA_∞-categories) in a filtered setting, encompassing the frameworks of Fukaya categories [5] and weakly curved A_∞-categories in the sense of Positselski [17]. Between two cA_∞-categories $\mathfrak{a}$ and $\mathfrak{b}$, we introduce a cA_∞-category $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$ of so-called qA_∞-functors in which the uncurved objects are precisely the cA_∞-functors from $\mathfrak{a}$ to $\mathfrak{b}$. The more general qA_∞-functors allow us to consider representable modules, a feature which is lost if one restricts attention to cA_∞-functors. We formulate a version of the Yoneda Lemma which shows every cA_∞-category to be homotopy equivalent to a curved dg category, in analogy with the uncurved situation. We also present a curved version of the bar-cobar adjunction.

1. Introduction

The theory of A_∞-categories is by now well-established, and furnishes a natural background for the development of non-commutative geometry [6], [11], [9], with applications ranging from Homological Mirror Symmetry [8] to the study of Fourier Mukai functors [21], [20]. From the homotopy perspective, dg categories and A_∞-categories are equivalent tools with their own particular advantages. On the one hand, dg categories are the simpler objects, sufficient to capture many important examples like categories of complexes over algebraic objects, but the development of their homotopy theory is involved and makes use of the Tabuada model structure to invert quasi-equivalences, leading on to the development of derived Morita theory [22], [23], [7]. On the other hand, A_∞-categories are more complex but the formalism allows for the construction of actual A_∞-functors outside the dg framework, avoiding the use of model categories. Over a field, both approaches are known to be equivalent. In particular, every A_∞-category is homotopy equivalent to a dg category. One of the useful features of the A_∞-framework is the possibility to construct natural functor categories which are themselves A_∞-categories [15]. For instance, this yields a natural way of looking at the Hochschild complex of an A_∞-category as the endomorphism algebra of the identity functor.

An A_∞-structure on a k-linear quiver $\mathfrak{a}$ can easily be defined as a special degree 2 element $m \in \mathbf{C}^2(\mathfrak{a})$, with $m \bullet m = 0$ for the dot product. From the point of view of deformation theory, it is quite unnatural to restrict

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attention to elements m with

$$0 = (m_0^A)_A \in \prod_{A \in \mathfrak{a}} \mathfrak{a}^2(A, A),$$

as is done in the classical definition of an A_∞-category. Indeed, a Hochschild cocycle ϕ may have a non-zero component ϕ_0 telling us to deform the component m_0 non-trivially. From the formal perspective, it is straightforward to define a *curved* A_∞-structure (cA_∞-structure) by the usual definition, only this time allowing $m^0 \neq 0$. However, as a consequence the component m^1 no longer squares to zero, and standard notions of cohomology disappear. Further, it has been observed by Kontsevich and others that if one introduces homotopy in the usual way, the curvature element of a cA_∞-algebra plays a very dominant role in the sense that under weak assumptions the algebra becomes homotopy equivalent to a cA_∞-algebra whose only non-zero component is the curvature element [18] [10]. As such, the general development of the theory of cA_∞-categories may be met with some skepticism, as it seems quite incompatible with the usual A_∞-theory. This being said, as cA_∞-categories naturally turn up, one should wonder how they fit into the framework of homological algebra. The present work is mainly inspired by two settings in which cA_∞-categories are often quite well behaved. Firstly, there is the important setting of Fukaya categories, which are constructed as cA_∞-categories over the Novikov ring. Here, the geometric phenomenon known as "bubbling" gives rise to curved objects, but in many cases these objects can be eventually avoided in the development of Floer cohomology. Secondly, in the deformation context, it often turns out that deformed cA_∞-categories can somehow be exchanged for dg or A_∞-categories in an appropriate sense [14], [13]. In both cases, this phenomenon is related to the curvature being "small" with respect to a natural filtration.

The goal of this work is the development of part of the basic theory of cA_∞-categories in a completely general filtered framework, sufficient to encompass the higher special cases. Precisely, we work over an ordered monoid $\mathbb{L}$ over which all algebraic structures are supposed to be filtered, and a commutative ground ring k. The neutral element 0 of the monoid $\mathbb{L}$ is supposed to be the smallest element for the order, and the filtrations of modules M are such that $\mathcal{F}^0 M = M$. Roughly speaking, the greater $l \in \mathbb{L}$ for which the curvature element is in $\mathcal{F}^l \mathfrak{a}^2(A, A)$, the smaller the curvature element is considered to be. Throughout, we check compatibility of our constructions with filtrations, in order that notions with traditional shortcomings may be useful in filtered settings. Many of our results are extensions of classical results in the A_∞ context, and our treatment is inspired by the treatment of A_∞-categories from [3], as well as by Positselski's work on cdg categories. In particular, apart from the Fukaya setup, Positselski's weakly curved A_∞-categories are a prime example fitting into our framework [17].

In §2, after presenting the necessary generalities on filtered structures we introduce the filtered Hochschild object of a filtered quiver, and we define filtered cA_∞-categories (Definition 2.47). In §3, we define cA_∞-functors (Definition 3.9) and the more primitive qA_∞-functors (Definition 3.11), where no structure compatibility is required. Even the definition of qA_∞-functors is subtle as the presence of curvature elements in the components of these functors poses convergence issues, that can be dealt with in the filtered setting. Our main result in this section is the construction of the functor category $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$ for cA_∞-categories $\mathfrak{a}$ and $\mathfrak{b}$, which is itself a cA_∞-category (Theorem 3.37). Moreover, the uncurved objects in this category are precisely the cA_∞-functors (Proposition 3.27). Further, we introduce cA_∞-homotopy equivalences between cA_∞-categories, and we argue that unlike in the unfiltered case, these need not trivialize the theory.

In §4, we present a cA_∞-version of the Yoneda Lemma. We consider the cdg subcategory $\mathsf{Mod}_{q\infty}(\mathfrak{a}) \subseteq \mathsf{qFun}(\mathfrak{a}^{\mathrm{op}}, \mathsf{PCom}(k))$ of strict qA_∞-functors (that is, functors without zero-th component) from $\mathfrak{a}^{\mathrm{op}}$ to the cA_∞-category $\mathsf{PCom}(k)$ of precomplexes of k-modules, and the further cdg subcategory $\mathsf{Rep}_{q\infty}(\mathfrak{a})$ of representable modules. In Theorem 4.15 we prove the existence of a "Yoneda" cA_∞-homotopy equivalence $Y : \mathfrak{a} \longrightarrow \mathsf{Rep}_{q\infty}(\mathfrak{a})$. In particular, every cA_∞-category is cA_∞-homotopy equivalent to a cdg category, and if the curvature component of an object in $\mathfrak{a}$ lies in $\mathcal{F}^l\mathfrak{a}$, then the curvature component of the corresponding representable module lies in $\mathcal{F}^l\mathsf{Rep}(\mathfrak{a})$.

As an application of Theorem 4.15, we define a "Yoneda tensor product" between cA_∞-categories through the cdg tensor product of the corresponding categories $\mathsf{Rep}(-)$. We also present an explicit construction of a tensor structure on the tensor quiver $\mathfrak{a} \otimes \mathfrak{b}$ in case one of the tensor factors is cdg, and show it to be cA_∞-homotopy equivalent to the Yoneda tensor product.

Finally, in §5, we extend the well known bar-cobar adjunction to the context of cA_∞-categories, recovering results by Nicolas [16] and Positselski [17] as particular cases.

This paper is part of a larger project, in which the aim is to understand the ways in which cA_∞-categories are to be viewed in non-commutative geometry. Here, we have focused on the development of basic ingredients like functor categories and the Yoneda Lemma. By keeping filtrations in the background, we avoid the mismatch between the curved and the uncurved world which exists in general. On the one hand, it is quite natural to expect functor categories between cA_∞-categories to be themselves cA_∞ rather than A_∞. On the other hand, it is standard practice in non-commutative algebraic geometry to study algebro-geometric objects through associated derived categories, which are obtained as the cohomology of suitable dg or A_∞-models. As such, it makes sense to consider Positselski's derived categories "of the second kind", and especially variants of the semi-derived category from [17] in the filtered case, as representing the "classical part" of cA_∞-categories. In future work, we want to extend the invariance results for qA_∞-categories of Proposition 3.51 in the direction of Morita theory, inspired by the deformation situation in [13] where a curved deformation can essentially be replaced by an uncurved one with an equivalent semi-derived category. We also want to investigate the relation of our work with an approach by Armstrong and Clarke, who propose an unfiltered notion of homotopy-equivalence based upon Morita invariance requirements [1].

2. Filtered cA_∞-categories

Let $\mathbb{L}$ be an ordered monoid and k an $\mathbb{L}$-filtered commutative ground ring. In this section, we introduce $\mathbb{L}$-filtered k-linear cA_∞-categories, where cA_∞ stands for *curved* A_∞. Roughly speaking, an $\mathbb{L}$-filtered cA_∞-category is a cA_∞-category in the category of $\mathbb{L}$-filtered k-modules. The unfiltered case is obtained by trivially filtering a commutative groundring k over $\mathbb{L} = \{0\}$.

After reviewing the basic theory of filtered modules, filtered algebraic structures and completion in §2.2 - §2.5, we introduce the filtered bar construction $B\mathfrak{a}$ and the filtered Hochschild object $\mathbf{C}(\mathfrak{a})$ of a filtered quiver $\mathfrak{a}$ in §2.6. In §2.8, we introduce filtered and complete formal cocategories, with $B\mathfrak{a}$ and its completion $\hat{B}\mathfrak{a}$ as main examples. In §2.9, a filtered cA_∞-structure on a filtered k-quiver $\mathfrak{a}$ is introduced as an element $m \in \mathbf{C}^2(\mathfrak{a})$ with $m\{m\} = 0$ for the first brace operation (the dot product). Equivalently, the natural coderivation on $B\mathfrak{a}$ (resp. the complete formal coderivation on $\hat{B}\mathfrak{a}$) determined by m is a codifferential, i.e squares to zero. The standout feature of a cA_∞-category in comparison with an A_∞-category is the possibly non-trivial curvature component

$$m_0 \in \prod_{A \in \mathfrak{a}} \mathfrak{a}^2(A, A).$$

If $m_0 \in \mathcal{F}^l C(\mathfrak{a})$ (where $\mathcal{F}^l$ denotes the l-th piece of the filtration for $l \in \mathbb{L}$), we call $\mathfrak{a}$ *l-curved.* In §2.10, we describe cA_∞-quotients by cA_∞-ideals. A situation of particular interest occurs if $\mathfrak{a}$ is l-curved, as we obtain an A_∞-quotient $\mathfrak{a}/\mathcal{F}^l\mathfrak{a}$. This quotient protects $\mathfrak{a}$ against some of the notoriously bad behaviour of cA_∞-categories, as we'll discuss further in §3.6.2.

2.1. **Ordered monoids.** Throughout, we will use the following definition:

Definition 2.1. An *ordered monoid* $\mathbb{L} = (\mathbb{L}, +, \leq)$ consists of a set $\mathbb{L}$ on which we have a commutative monoid structure $(\mathbb{L}, +)$ and a partial order $(\mathbb{L}, \leq)$ such that the following conditions are fulfilled:

(1) for $a, b, c \in \mathbb{L}$ with $a \leq b$, we have $a + c \leq b + c$.
(2) the neutral element $0 \in \mathbb{L}$ for $(\mathbb{L}, +)$ is the smallest element for $(\mathbb{L}, \leq)$.

For $n \in \mathbb{N}$ and $a \in \mathbb{L}$, we denote $na = \sum_{i=1}^n a$. Here, we interpret $0a = 0$. Later on, we will make use of the following property:

Definition 2.2. The ordered monoid $\mathbb{L}$ is *archimedean* if the following property holds: for all $0 < a$ and $0 < b$ in $\mathbb{L}$, there exists $n \in \mathbb{N}$ with $b \leq na$.

Example 2.3. The set $\mathbb{L} = \{0\}$ is endowed with a unique (archimedean) ordered monoid structure, which we call the trivial ordered monoid.

Example 2.4. If $\mathbb{L}$ has a largest element ∞, we necessarily have $l + \infty = \infty$, $\infty + l = \infty$ and $\infty + \infty = \infty$ for $l \in \mathbb{L}$. Conversely, to an arbitrary $\mathbb{L}$ we can adjoin a largest element ∞, and we can uniquely endow $\mathbb{L}^\infty = \mathbb{L} \cup \{\infty\}$ with the structure of ordered monoid. We thus obtain the (archimedean) ordered monoid $\mathbb{S} = \{0\}^\infty = \{0, \infty\}$ for $\mathbb{L} = \{0\}$ as in Example 2.3.

Example 2.5. The natural numbers $\mathbb{N}$, the non-negative rational numbers $\mathbb{Q}^+$, and the non-negative real numbers $\mathbb{R}^+$ are archimedean ordered monoids for the usual summation $+$ and the usual order $\leq$.

2.2. **Filtered modules.** Let k be a commutative ground ring. Since we are mainly interested in $\mathbb{Z}$-graded objects, our starting point is the category $\mathsf{Mod}(k)$ of $\mathbb{Z}$-graded k-modules $(M^n)_n$ and $\mathbb{Z}$-graded morphisms. These will be simply referred to as k-modules and morphisms. Ordinary k-modules and morphisms are considered as $\mathbb{Z}$-graded by placing them in degree 0. The category $\mathsf{Mod}(k)$ is symmetric monoidal closed with

$$(M \otimes_k N)^n = \oplus_{p+q=n} M^p \otimes_k N^q$$

and

$$\mathrm{Hom}_k(M, N)^n = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_k(M^i, N^{i+n}).$$

Throughout, we adopt the sign convention from [12, §2.1].

Let $(\mathbb{L}, +, \leq)$ be a commutative ordered monoid as in Definition 2.1.

Definition 2.6. An $\mathbb{L}$-*filtered k-module* (or simply *filtered k-module*) is a k-module M together with, for every $l \in \mathbb{L}$, a submodule $\mathcal{F}^l M \subseteq M$ such that $\mathcal{F}^0 M = M$ and $l' \leq l$ implies $\mathcal{F}^l M \subseteq \mathcal{F}^{l'} M$.

Consider filtered k-modules M and N. The tensor product $M \otimes_k N$ is naturally filtered with

$$\mathcal{F}^l(M \otimes_k N) = \mathrm{Im}(\oplus_{p+q \geq l} \mathcal{F}^p M \otimes_k \mathcal{F}^q N \longrightarrow M \otimes_k N). \tag{1}$$

More generally, for filtered k-modules M_i, $1 \leq i \leq n$, the tensor product $\otimes_{i=1}^n M_i$ is filtered with

$$\mathcal{F}^l(\otimes_{i=1}^n M_i) = \mathrm{Im}(\oplus_{\sum_i l_i = l} \otimes_{i=1}^n \mathcal{F}^{l_i} M_i \longrightarrow \otimes_{i=1}^n M_i). \tag{2}$$

Definition 2.7. An $\mathbb{L}$-*filtered k-linear morphism* (or simply a *filtered k-morphism*) from M to N is a k-linear morphism $f : M \longrightarrow N$ with $f(\mathcal{F}^l M) \subseteq \mathcal{F}^l N$ for each $l \in \mathbb{L}$.

Filtered k-morphisms constitute a filtered module ${}_k[M, N]$ with

$$\mathcal{F}^l({}_k[M, N]) = \{f \in \mathrm{Hom}_k(M, N) \mid f(\mathcal{F}^p M) \subseteq \mathcal{F}^{p+l} N\}. \tag{3}$$

Remark 2.8. Note that unlike for the tensor product $M \otimes_k N$, as a k-module ${}_k[M, N]$ is *not* equal to the module $\mathrm{Hom}_k(M, N)$ of k-linear morphisms from M to N.

By definition, ${}_k[\otimes_{i=1}^n M_i, M]$ contains all $f \in \mathrm{Hom}_k(\otimes_{i=1}^n M_i, M)$ with

$$f(\mathcal{F}^{l_1} M_1, \dots, \mathcal{F}^{l_n} M_n) \subseteq \mathcal{F}^{l_1 + \dots + l_n} M. \tag{4}$$

We thus obtain a symmetric monoidal closed category $\mathsf{Mod}_{\mathbb{L}}(k)$ of filtered k-modules and filtered k-morphisms, for which the (monoidal) forgetful functor $\mathsf{Mod}_{\mathbb{L}}(k) \longrightarrow \mathsf{Mod}(k)$ has both adjoints. The (monoidal) left adjoint is obtained by endowing a k-module M with the *discrete* filtration with $\mathcal{F}^0 M = M$ and $\mathcal{F}^l M = 0$ for $l \neq 0$. The (monoidal) right adjoint is obtained by endowing a k-module M with the *trivial* filtration with $\mathcal{F}^l M = M$ for all l.

Example 2.9. Let $\mathbb{L} = \{0\}$ be the trivial ordered monoid. Then $\mathbb{L}$-filtered k-modules simply correspond to k-modules, and we recover the usual notions of k-linear morphisms, tensor modules and Hom-modules.

Example 2.10. Let $\mathbb{L}$ be an orderer monoid with $\mathbb{L}^\infty = \mathbb{L} \cup \{\infty\}$ as in Example 2.4. Every $\mathbb{L}$-filtered k-module becomes $\mathbb{L}^\infty$-filtered by putting $\mathcal{F}^\infty M = 0$. With this definition, by k-linearity, an $\mathbb{L}$-filtered k-morphism automatically becomes $\mathbb{L}^\infty$-filtered. The ordered monoid $\mathbb{S} = \{0\}^\infty$ can thus be used to endow any k-module M with the $\mathbb{S}$-filtration $\mathcal{F}^0 M = M$ and $\mathcal{F}^\infty M = 0$.

2.3. **Filtered structures.** Let S be a commutative ground ring and let $\mathbb{L}$ be as before. The usual algebraic structures can be defined with respect to the symmetric monoidal category $\mathsf{Mod}_{\mathbb{L}}(S)$.

Definition 2.11.
(1) A *filtered S-algebra* is a filtered S-module k with a filtered S-morphism $m : k \otimes_S k \longrightarrow k$ satisfying the associativity relation $m(m \otimes 1) = m(1 \otimes m)$. A filtered $\mathbb{Z}$-algebra is called a *filtered ring.*
(2) A filtered S-algebra k is *unital* with *unit* $1_k \in k$ if $m(1_k, x) = x = m(x, 1_k)$ for all $x \in k$.
(3) For a filtered S-algebra k, a *filtered (left) k-module* is a filtered S-module M with a filtered S-morphism $\rho : k \otimes_S M \longrightarrow M$ satisfying $\rho(1 \otimes \rho) = \rho(m \otimes 1)$.
(4) A filtered left k-module is *unital* if $\rho(1_k, m) = m$ for all $m \in M$.

From now on, we will always include unitality in the notions of filtered algebras and modules unless otherwise stated. As usual, the operations like m and ρ are simply denoted by juxtaposition.

For a filtered S-algebra k, we obtain a category $\mathsf{Mod}_{\mathbb{L}}(k)$ of filtered left k-modules with filtered k-linear morphisms (k-morphism for short). As usual, this category is independent of S (we may take $S = \mathbb{Z}$ and consider k as a filtered ring).

Remark 2.12.
(1) Let k be a filtered S-algebra and M a filtered left k-module. By definition, we have $\rho(\mathcal{F}^0 k, \mathcal{F}^n M) \subseteq \mathcal{F}^n M$ so each $\mathcal{F}^n M$ is an $\mathcal{F}^0 k$-module.
(2) We can consider S with the discrete filtration as a filtered S-algebra, for which the modules are precisely the filtered S-modules. Thus, a unit for k corresponds to a filtered map $S \longrightarrow k$. Suppose $1_k \in \mathcal{F}^l k$. Then for all $x \in k$, we have $x = 1_k x \in \mathcal{F}^l k$. Thus, we conclude that necessarliy $\mathcal{F}^l k = k$. The filtration of k is called *proper* if $1_k \in \mathcal{F}^l k$ implies $l = 0$. In the sequel, we will assume that all filtered S-algebras are properly filtered.

2.4. **Filtered modules over a filtered ring.** Suppose k is a commutative filtered ring. Then we can repeat over k the constructions we performed in the previous paragraph. Precisely, for filtered k-modules M and N, we obtain the tensor product $M \otimes_k N$ with filtration given by (1). More generally, for filtered k-modules M_i, $1 \leq i \leq n$, the tensor product $\otimes_{i=1}^n M_i$ is filtered with

$$\mathcal{F}^l(\otimes_{i=1}^n M_i) = \mathrm{Im}(\oplus_{\sum_i l_i = l} \otimes_{i=1}^n \mathcal{F}^{l_i} M_i \longrightarrow \otimes_{i=1}^n M_i). \tag{5}$$

The category $\mathsf{Mod}_{\mathbb{L}}(k)$ is symmetric monoidal and we have a (monoidal) forgetful functor $\mathsf{Mod}_{\mathbb{L}}(k) \longrightarrow \mathsf{Mod}(k)$. The (monoidal) left adjoint is obtained by endowing a k-module M with the *canonical* filtration with

$$\mathcal{F}^l M = \mathcal{F}^l k \cdot M = \{\sum_{i=0}^{n} \alpha_i m_i \mid \alpha_i \in \mathcal{F}^l k, m_i \in M\}.$$

Obviously, k is itself endowed with the canonical filtration. If k-modules M_i, $1 \leq i \leq n$, are endowed with the canonical filtration, then by (5), $\otimes_{i=1}^n M_i$ is also endowed with the canonical filtration.

Filtered k-morphisms from M to N are defined as in Definition 2.7, and they constitute a filtered k-module ${}_k[M,N]$ with filtration given by (3). By definition, $\mathcal{F}^l({}_k[\otimes_{i=1}^n M_i, M])$ contains all $f \in \mathrm{Hom}_k(\otimes_{i=1}^n M_i, M)$ with

$$f(\mathcal{F}^{l_1} M_1, \ldots, \mathcal{F}^{l_n} M_n) \subseteq \mathcal{F}^{l+l_1+\cdots+l_n} M. \tag{6}$$

Remark 2.13. If the k-module M is canonically filtered, have ${}_k[M,N] = \mathrm{Hom}_k(M,N)$.

Consider a family $(M_i)_i$ of filtered k-modules. The sum $\oplus_i M_i$ and the product $\prod_i M_i$ are naturally filtered modules with $\mathcal{F}^l(\oplus_i M_i) = \oplus_i \mathcal{F}^l M_i$ and $\mathcal{F}^l(\prod_i M_i) = \prod_i \mathcal{F}^l M_i$. More generally, we can endow an arbitrary limit $\mathrm{lim}_i M_i$ with the filtration $\mathcal{F}^l \mathrm{lim}_i M_i = \mathrm{lim}_i \mathcal{F}^l M_i$, and directed colimits being exact, we can endow a directed colimit $\mathrm{colim}_i M_i$ with the filtration $\mathcal{F}^l \mathrm{colim}_i M_i = \mathrm{colim}_i \mathcal{F}^l M_i$. If the modules M_i are canonically filtered, all these constructions are endowed with the canonical filtrations as well.

A submodule $M \subseteq N$ with both M and N filtered is called a *filtered submodule* if $\mathcal{F}^l M \subseteq \mathcal{F}^l N$, i.e. if the inclusion is a filtered morphism. In general, there is no natural way to filter the quotient of a filtered module by a filtered submodule. However, let N be a filtered k-module and $M \subseteq N$ a k-submodule. Then we obtain the *pullback filtration* on M with $\mathcal{F}^l M = M \cap \mathcal{F}^l N$ and we obtain the *quotient filtration* on N/M with $\mathcal{F}^l(N/M) = \mathcal{F}^l N / M \cap \mathcal{F}^l N$.

For a subset $A \subseteq k$ and $n \in \mathbb{N}$, we consider the subset $\sqrt[n]{A} = \{x \in k \mid x^n \in A\} \subseteq k$. If A is an ideal in k, then $A \subseteq \sqrt[n]{A}$. In this case, the *radical* of A, $\mathrm{rad}(A) = \cup_{n \in \mathbb{N}} \sqrt[n]{A}$ is an ideal and A is called *radical* if $A = \mathrm{rad}(A)$. Note that $\mathcal{F}^l k$ is an ideal for $l \in \mathbb{L}$.

The following notion will be used later on:

Definition 2.14. The filtered ring k is called *radically filtered* if the following condition holds: for every $0 < l \in \mathbb{L}$ and $n \in \mathbb{N}$, there exists $0 < l' \in \mathbb{L}$ with

$$\sqrt[n]{\mathcal{F}^l k} \subseteq \mathcal{F}^{l'} k. \tag{7}$$

Remark 2.15. If $l_0 = \min\{l \in \mathbb{L} \mid 0 < l\}$ exists in $\mathbb{L}$, then k is radically filtered if and only if the ideal $\mathcal{F}^{l^0} k$ is radical.

Lemma 2.16. *Suppose k is radically filtered and let l, n, l' be as in* (7). *Let M be a free k-module endowed with the canonical filtration. If $x \in M$ satisfies $x^{\otimes n} \in \mathcal{F}^l(M^{\otimes n})$, then $x \in \mathcal{F}^{l'} M$.*

Proof. For $M = k^I$, the canonical filtration is given by $\mathcal{F}^l M = \oplus_I \mathcal{F}^l k$. We have $\mathcal{F}^\lambda M = \oplus_I \mathcal{F}^\lambda k$ and we have $M^{\otimes n} = k^{I^n}$ and $\mathcal{F}^\lambda M^{\otimes n} = \oplus_{I^n} \mathcal{F}^\lambda k$. Write $x = (x_i)_{i \in I} \in k^I$. For $x^{\otimes n}$, we have $(x^{\otimes n})_{(i,\dots,i)} = (x_i)^n$. If $x^{\otimes n} \in \mathcal{F}^l M^{\otimes n}$, we thus have $(x_i)^n \in \mathcal{F}^l k$ and $x_i \in \sqrt[n]{\mathcal{F}^l k} \subseteq \mathcal{F}^{l'} k$. It follows that $x \in \mathcal{F}^{l'} M$. $\square$

Example 2.17. Let k be a ring and let $\mathbb{S} = \{0, \infty\}$ be as in Example 2.4. Then k becomes $\mathbb{S}$-filtered with $\mathcal{F}^\infty k = 0$ and we can endow every k-module M with the canonical discrete $\mathbb{S}$-filtration for which $\mathcal{F}^\infty M = 0$, for which M becomes a filtered k-module over the filtered ring k.

By Remark 2.15, k is radically filtered if and only if $\{0\} \subseteq k$ is a radical ideal.

Example 2.18. Let k be a ring and $I \subseteq k$ an ideal. Take $\mathbb{L} = \mathbb{N}$. Then k becomes a filtered ring by putting $\mathcal{F}^n k = I^n k$, the so called *I-adic filtration.* A filtered k-module is a k-module M with a filtration $\mathcal{F}^n M$ such that $I^m \mathcal{F}^n \subseteq \mathcal{F}^{n+m}$. In particular, every k-module M can be endowed with the canonical I-adic filtration by putting $\mathcal{F}^n M = I^n M$.

By Remark 2.15, k is radically filtered if and only if $I \subseteq k$ is a radical ideal.

Example 2.19. Let S be a commutative ring. The *monoid S-algebra* $S[\mathbb{L}]$ is the free S-module on generators t^λ for $\lambda \in \mathbb{L}$, endowed with the multiplication $(at^\lambda)(bt^{\lambda'}) = abt^{\lambda+\lambda'}$. The ring $S[\mathbb{L}]$ becomes a filtered S-algebra with $\mathcal{F}^l S[\mathbb{L}]$ given by the S-submodule generated by $\{t^\lambda \mid \lambda \geq l\}$.

Taking $\mathbb{L} = \mathbb{N}$, we obtain $S[\mathbb{N}] = S[t]$, the ring of polynomials with coefficients in S, with $\mathcal{F}^l S[t] = t^l S[t]$.

Example 2.20. Let S be a commutative ring. The ring $S[[\mathbb{L}]]$ is the ring of formal expressions $\sum_{\lambda \in \mathbb{L}} a_\lambda t^\lambda$ with $a_\lambda \in S$ and t a formal parameter, subject to the following condition:

($*$) for each $\lambda \in \mathbb{L}$, the number of $\kappa \not\geq \lambda$ with $a_\kappa \neq 0$ is finite.

The operations on $S[[\mathbb{L}]]$ are given by

$$(\sum_\lambda a_\lambda t^\lambda) + (\sum_\lambda b_\lambda t^\lambda) = \sum_\lambda (a_\lambda + b_\lambda) t^\lambda$$

and

$$(\sum_\lambda a_\lambda t^\lambda)(\sum_\lambda b_\lambda t^\lambda) = \sum_\lambda (\sum_{\lambda' + \lambda'' = \lambda} a_{\lambda'} b_{\lambda''}) t^\lambda.$$

Note that for the second expression to make sense, we have to ensure that for given λ, the number of pairs (λ', λ'') with $\lambda' + \lambda'' = \lambda$ and $a_{\lambda'} b_{\lambda''} \neq 0$, is finite. But from $0 \leq \lambda''$ we obtain $\lambda' \leq \lambda' + \lambda'' = \lambda$ and similarly $\lambda'' \leq \lambda$. It thus follows from condition ($*$) on the coefficients that the inner sum is finite, and the resulting expression is seen to satisfy ($*$).

The ring $S[[\mathbb{L}]]$ is endowed with the filtration

$$\mathcal{F}^l S[[\mathbb{L}]] = \{\sum_\lambda a_\lambda t^\lambda \mid \lambda \not\geq l \implies a_\lambda = 0\}.$$

It is seen to be a filtered ring (in fact, a filtered S-algebra). Suppose for all $l \leq \lambda$ in $\mathbb{L}$, there exists $l' \in \mathbb{L}$ with $\lambda = l + l'$. Then we have $\mathcal{F}^l S[[\mathbb{L}]] = t^l S[[\mathbb{L}]]$.

Example 2.21. Let S be a commutative ring and put $k = S[[t]]$. Take $\mathbb{L} = \mathbb{N}$. We have $S[[\mathbb{N}]] = S[[t]]$, the ring of formal power series. The filtration described in Example 2.20 is precisely the (t)-adic filtration described in Example 2.18.

Example 2.22. Take $\mathbb{L} = \mathbb{R}^+$ in Example 2.20. Consider $x = \sum_{\lambda\in\mathbb{L}} a_\lambda t^\lambda \in k = S[[\mathbb{R}^+]]$. It follows from condition $(*)$ that the elements λ for which $a_\lambda \neq 0$ form a countable closed subset of $\mathbb{R}^+$ and we can rewrite $x = \sum_{n=0}^{\infty} a_{\lambda_n} t^{\lambda^n}$ with $\lim_{n\to\infty}\lambda_n = \infty$. Here, it may of course happen that $a_{\lambda_n} = 0$ for $n \geq n_0$ for a certain n_0. Also, we have $\mathcal{F}^l k = t^l k$.

Let us now show that $k = S[[\mathbb{R}^+]]$ is radically filtered as soon as $\{0\}$ is a radical ideal in S. Consider $x = \sum_{\lambda\in\mathbb{L}} a_\lambda t^\lambda \in k$ and let $\rho = \min\{\lambda \in \mathbb{L} \mid a_\lambda \neq 0\}$. We have $x = t^\rho y$ for some $y = \sum_{\lambda\in\mathbb{L}} b_\lambda t^\lambda \in k$ with $b_0 \neq 0$. Hence, $x^n = t^{n\rho} y^n$ and $y^n = \sum_{\lambda\in\mathbb{L}} c_\lambda t^\lambda$ with $c_0 = b_0^n \neq 0$. Hence, if $x^n \in \mathcal{F}^l k = t^l k$, we deduce that $l \leq \rho n$ and $\frac{l}{n} \leq \rho$. It follows that $x \in \mathcal{F}^{\frac{l}{n}} k$. Thus, in (7), we can take $l' = \frac{l}{n}$.

In [4], Fukaya defines the *universal Novikov ring* as $\Lambda_{nov,0} = \mathbb{Q}[\xi][[\mathbb{R}^+]]$ where $S = \mathbb{Q}[\xi]$ is the $\mathbb{Z}$-graded polynomial ring with $\deg(\xi) = 2$ (see also [5]).

2.5. **Completion.** Let k be a commutative $\mathbb{L}$-filtered ring and let M be a k-module. We can define the *completion* $\hat{M}$ of M in a purely algebraic way as

$$\hat{M} = \lim_{\lambda\in\mathbb{L}} M/\mathcal{F}^\lambda M.$$

with the pointwise sum and k-action. The quotients $M/\mathcal{F}^\lambda M$ are naturally filtered by $\mathcal{F}^l(M/\mathcal{F}^\lambda M) = \mathcal{F}^l M/\mathcal{F}^\lambda M \cap \mathcal{F}^l M$ and $\lim_\lambda M/\mathcal{F}^\lambda M$ is filtered by

$$\mathcal{F}^l \hat{M} = \lim_\lambda \mathcal{F}^l M/\mathcal{F}^\lambda M \cap \mathcal{F}^l M.$$

Note that for $\lambda \leq l$, we have $\mathcal{F}^l M/\mathcal{F}^\lambda M \cap \mathcal{F}^l M = 0$.

There is a canonical filtered k-morphism $M \longrightarrow \hat{M}$. The module M is *complete* provided that this morphism is an isomorphism. A filtered morphism $M \longrightarrow N$ naturally induces a filtered morphism $\hat{M} \longrightarrow \hat{N}$.

Algebraic structures on M can be carried over to $\hat{M}$. For instance, if R is a filtered ring, $\hat{R}$ becomes a filtered ring with multiplication given by $(x_\lambda)(y_\lambda) = (x_\lambda y_\lambda)$ with $x_\lambda y_\lambda$ defined using representatives in M.

Example 2.23. The filtered S-algebra $S[[\mathbb{L}]]$ from Example 2.20 can isomorphically be described as the completion of the filtered monoid S-algebra $S[\mathbb{L}]$ from Example 2.19. This generalizes the well known fact that $S[[t]]$ is obtained as the completion of $S[t]$ with respect to the t-adic filtration.

2.6. **The filtered Hochschild object of a filtered quiver.** Let $(\mathbb{L}, +, \leq)$ be an ordered monoid as before and let k be a commutative $\mathbb{L}$-filtered ground ring. Unless otherwise stated, all constructions are over k. Since we will work in a multi-object setup, our fundamental objects are quivers rather than modules.

Definition 2.24. An $\mathbb{L}$-*filtered k-quiver* (or simply *(filtered) k-quiver*) $\mathfrak{a}$ consists of a set of objects $\mathrm{Ob}(\mathfrak{a})$ and for all $A, A' \in \mathrm{Ob}(\mathfrak{a})$, a filtered k-module $\mathfrak{a}(A, A') \in \mathsf{Mod}_{\mathbb{L}}(k)$.

Definition 2.25. Consider $\mathbb{L}$-filtered k-quivers $\mathfrak{a}$ and $\mathfrak{b}$. An *$\mathbb{L}$-filtered k-morphism* (or simply a *filtered k-morphism*) $f : \mathfrak{a} \longrightarrow \mathfrak{b}$ is given by an underlying map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$ and for all $A, B \in \mathrm{Ob}(\mathfrak{a})$ an $\mathbb{L}$-filtered k-morphism

$$f_{A,B} : \mathfrak{a}(A, B) \longrightarrow \mathfrak{b}(f(A), f(B)).$$

If the maps $f_{A,B}$ are given by inclusions of filtered submodules (or, more generally, by filtered monomorphisms), we write $\mathfrak{a} \subseteq \mathfrak{b}$ and we call $\mathfrak{a}$ a *subquiver* of $\mathfrak{b}$.

Remark 2.26. Taking $\mathbb{L} = \{0\}$, we recover the standard notions of k-quivers and morphisms of k-quivers (see [2]).

For a filtered k-quiver $\mathfrak{a}$, the *completion* $\hat{\mathfrak{a}}$ is obtained by completing all the individual k-modules $\mathfrak{a}(A, A')$ in the sense of §2.5, and $\mathfrak{a}$ is called *complete* if the resulting morphism $\mathfrak{a} \longrightarrow \hat{\mathfrak{a}}$ is an isomorphism.

Consider filtered k-quivers $\mathfrak{a}$ and $\mathfrak{b}$ and a map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$. We define the filtered k-module

$${}_k[\mathfrak{a}, \mathfrak{b}]_f = \prod_{A,A' \in \mathfrak{a}} {}_k[\mathfrak{a}(A, A'), \mathfrak{b}(f(A), f(A'))].$$

If $\mathrm{Ob}(\mathfrak{a}) = \mathrm{Ob}(\mathfrak{b})$, we define the *tensor product* $\mathfrak{a} \otimes_k \mathfrak{b}$ as the filtered k-quiver with the same set of objects and

$$\mathfrak{a} \otimes_k \mathfrak{b}(A, A') = \bigoplus_{A'' \in \mathfrak{a}} \mathfrak{a}(A'', A') \otimes_k \mathfrak{b}(A, A'').$$

In the sequel, all constructions will be over k unless stated otherwise, so for legibility we will use unadorned notations for tensor products ($\otimes = \otimes_k$) and filtered morphisms ($[-,-] =_k [-,-]$).

Remark 2.27. The tensor product of quivers with the same set of objects which we just defined and which will be used throughout the paper should not be confused with another standard tensor product that exists between arbitrary quivers, and that produces a new quiver with the product of the two object sets as new object set.

We define $k\mathrm{Ob}(\mathfrak{a})$ to be the filtered k-quiver with the same object set as $\mathfrak{a}$ and

$$k\mathrm{Ob}(\mathfrak{a})(A, A') = \begin{cases} k & \text{if } A = A' \\ 0 & \text{else.} \end{cases}$$

Let $f : \mathfrak{a} \longrightarrow \mathfrak{b}$ be a filtered morphism of quivers. We define the filtered k-morphism

$$k\mathrm{Ob}(f) : k\mathrm{Ob}(\mathfrak{a}) \longrightarrow k\mathrm{Ob}(\mathfrak{b})$$

by the same underlying map $\mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$ as f and $k\mathrm{Ob}(f)_{(A,A)} : k \longrightarrow k$ equal to the identity morphism on k.

Clearly, $k\mathrm{Ob}(\mathfrak{a})$ is the unit with respect to the tensor product, so we put $\mathfrak{a}^{\otimes 0} = k\mathrm{Ob}(\mathfrak{a})$.

The *tensor k-quiver* $T(\mathfrak{a})$ is the filtered k-quiver

$$T(\mathfrak{a}) = \oplus_{n \geq 0} \mathfrak{a}^{\otimes n}$$

For filtered k-quivers $\mathfrak{a}$ and $\mathfrak{b}$, and a map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$, we put $[T(\mathfrak{a}), \mathfrak{b}]_{f,n} = [\mathfrak{a}^{\otimes n}, \mathfrak{b}]_f$ which is given by

$$\prod_{A_0,\dots,A_n\in\mathfrak{a}} [\mathfrak{a}(A_{n-1}, A_n) \otimes \cdots \otimes \mathfrak{a}(A_0, A_1), \mathfrak{b}(f(A_0), f(A_n))].$$

By definition of the filtered tensor product,

$$\phi \in [\mathfrak{a}(A_{n-1}, A_n) \otimes \cdots \otimes \mathfrak{a}(A_0, A_1), \mathfrak{b}(f(A_0), f(A_n))]$$

is given by $\phi \in \mathrm{Hom}_k(\mathfrak{a}(A_{n-1}, A_n) \otimes \cdots \otimes \mathfrak{a}(A_0, A_1), \mathfrak{b}(f(A_0), f(A_n)))$ with

$$\phi(\mathcal{F}^{l_n}\mathfrak{a}(A_{n-1}, A_n), \dots, \mathcal{F}^{l_1}\mathfrak{a}(A_0, A_1)) \subseteq \mathcal{F}^{l_1+\cdots+l_n}\mathfrak{b}(f(A_0), f(A_n)).$$

Remark 2.28. The *zero part* is given by

$$[T(\mathfrak{a}), \mathfrak{b}]_{f,0} = \prod_{A\in\mathfrak{a}} [k, \mathfrak{b}(f(A), f(A))] = \prod_{A\in\mathfrak{a}} \mathfrak{b}(f(A), f(A))$$

where we have used Remark 2.13.

We thus obtain

$$[T(\mathfrak{a}), \mathfrak{b}]_f = \prod_{n\geq 0} [T(\mathfrak{a}), \mathfrak{b}]_{f,n}$$

which is endowed with a natural projection

$$p_0 : [T(\mathfrak{a}), \mathfrak{b}]_f \longrightarrow [T(\mathfrak{a}), \mathfrak{b}]_{f,0}$$

onto the zero part. Suppose an element $J_f \in [\mathfrak{a}, \mathfrak{b}]_f$ has been chosen.

Consider another filtered k-quiver $\mathfrak{c}$ and map $g : \mathrm{Ob}(\mathfrak{b}) \longrightarrow \mathrm{Ob}(\mathfrak{c})$. We obtain *brace-compositions*

$$[T(\mathfrak{b}), \mathfrak{c}]_{g,n} \otimes_k [T(\mathfrak{a}), \mathfrak{b}]_{f,n_1} \otimes_k \cdots \otimes_k [T(\mathfrak{a}), \mathfrak{b}]_{f,n_k} \longrightarrow [T(\mathfrak{a}), \mathfrak{c}]_{gf,n-k+n_1+\cdots+n_k}$$

with

$$\phi\{\phi_1, \dots, \phi_k\} = \sum \phi(J_f \otimes \cdots \otimes \phi_1 \otimes J_f \otimes \cdots \otimes \phi_k \otimes J_f \otimes \cdots \otimes J_f). \tag{8}$$

Remark 2.29. The morphisms occuring in $\phi\{\phi_1, \dots, \phi_k\}$ are readily seen to be filtered, since the morphisms occuring in ϕ, ϕ_j for all $1 \leq j \leq k$ and J_f are filtered.

Remark 2.30. The element $J_f \in [\mathfrak{a}, \mathfrak{b}]_f$ should be thought of as a kind of identity map from $\mathfrak{a}$ to $\mathfrak{b}$, offering a "trivial" way to transport elements from $\mathfrak{a}$ to $\mathfrak{b}$.

Remark 2.31. Let filtered k-quivers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ and maps

$$\mathrm{Ob}(\mathfrak{a}) \xrightarrow[f]{} \mathrm{Ob}(\mathfrak{b}) \xrightarrow[g]{} \mathrm{Ob}(\mathfrak{c}) \xrightarrow[h]{} \mathrm{Ob}(\mathfrak{d})$$

be fixed and suppose brace operations are defined with respect to (compositions) of these maps. Consider elements $\phi \in [T(\mathfrak{c}), \mathfrak{d}]_h$, $\phi_i \in [T(\mathfrak{b}), \mathfrak{c}]_g$, $\psi_j \in [T(\mathfrak{a}), \mathfrak{b}]_f$. If we use identity-like element $J_f \in [\mathfrak{a}, \mathfrak{b}]_f$, $J_g \in [\mathfrak{b}, \mathfrak{c}]_g$ and $J_{gf} = J_g J_f \in [\mathfrak{a}, \mathfrak{c}]_{gf}$, then the operations (8) satisfy the brace-type axiom (see [12, Definition 2.1]), i.e.

$$\phi\{\phi_1, \dots, \phi_m\}\{\psi_1, \dots, \psi_n\}$$
$$= \sum (-1)^\alpha \phi\{J_g\psi_1, \dots, \phi_1\{\psi_{i_1}, \dots\}, J_g\psi_{j_1}, \dots, \phi_m\{\psi_{i_m}, \dots\}, J_g\psi_{j_m}, \dots, J_g\psi_n\}$$

where $\alpha = \sum_{k=1}^m |\phi_k| \sum_{l=1}^{i_k-1} |\psi_l|$.

Let $\Sigma\mathfrak{a}$ denote the shift of $\mathfrak{a}$, i.e. the filtered k-quiver with $\Sigma\mathfrak{a}(A,A')^i = \mathfrak{a}(A,A')^{i+1}$.

Definition 2.32. We define the *bar construction* of a filtered k-quiver $\mathfrak{a}$ to be the filtered k-quiver

$$B\mathfrak{a} = T(\Sigma\mathfrak{a}).$$

The completion of $B\mathfrak{a}$ with respect to its filtration is denoted by $\hat{B}\mathfrak{a}$.

Definition 2.33. Consider filtered k-quivers $\mathfrak{a}$ and $\mathfrak{b}$ and a map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$. We define

$$\mathbf{C}_{br}(\mathfrak{a},\mathfrak{b})_f = [B\mathfrak{a}, \Sigma\mathfrak{b}]_f$$

and associate to it the *Hochschild object* of $\mathfrak{a}$ and $\mathfrak{b}$ with respect to f, given by

$$\mathbf{C}(\mathfrak{a},\mathfrak{b})_f = \Sigma^{-1}\mathbf{C}_{br}(\mathfrak{a},\mathfrak{b})_f.$$

Remark 2.34. We denote $\mathbf{C}_{br}(\mathfrak{a}) = \mathbf{C}_{br}(\mathfrak{a},\mathfrak{a})_{1_\mathfrak{a}}$ and $\mathbf{C}(\mathfrak{a}) = \Sigma^{-1}\mathbf{C}_{br}(\mathfrak{a})$. The former is a brace algebra for the standard brace compositions, and we will freely consider the transfered compositions on the latter as well.

Considering $(\mathbf{C}(\mathfrak{a},\mathfrak{b})_f)_0 = \Sigma^{-1}[B\mathfrak{a}, \Sigma\mathfrak{b}]_{f,0}$, we obtain the projection

$$p_0 : \mathbf{C}(\mathfrak{a},\mathfrak{b})_f \longrightarrow (\mathbf{C}(\mathfrak{a},\mathfrak{b})_f)_0$$

onto the zero part.

2.7. **Tensor convergent collections.** In this section, we investigate the notion of tensor convergence which will be crucial later on in §3.1.

Definition 2.35. Let $\mathfrak{a}$ be a filtered quiver, X a set and $f : X \longrightarrow \mathrm{Ob}(\mathfrak{a})$ a map. A collection of elements $(\alpha_x)_{x\in X}$ with $\alpha_x \in \mathfrak{a}(f(x), f(x))$ is called *tensor convergent* if for every $l \in \mathbb{L}$ there exists $n \in \mathbb{N}$ such that for every element

$$\gamma = \beta_k \otimes \cdots \otimes \beta_1 \in \mathfrak{a}(A_{k-1}, A_k) \otimes \cdots \otimes \mathfrak{a}(A_0, A_1)$$

for which there exist n different indices $i_1, \dots i_n$ and elements $x_1, \dots, x_n \in X$ with $\beta_{i_m} = \alpha_{x_m}$, we have that $\gamma \in \mathcal{F}^l(\mathfrak{a}(A_{k-1}, A_k) \otimes \cdots \otimes \mathfrak{a}(A_0, A_1))$.

Proposition 2.36. *Let $\mathfrak{a}$ be a filtered quiver, X a set, $f : X \longrightarrow \mathrm{Ob}(\mathfrak{a})$ a map, and $(\alpha_x)_{x\in X}$ a collection of elements with $\alpha_x \in \mathfrak{a}(f(x), f(x))$. Suppose $\mathbb{L}$ is archimedean. Suppose there exists $0 < \lambda \in \mathbb{L}$ with $\alpha_x \in \mathcal{F}^\lambda(\mathfrak{a}(f(x), f(x))$ for all $x \in X$. Then the collection $(\alpha_x)_{x\in X}$ is tensor convergent.*

Proof. For $0 \leq l \in \mathbb{L}$, take $N_l \in \mathbb{N}$ such that $l \leq \sum_{i=1}^{N_l} \lambda$. For $n \geq N_l$, we have $\gamma \in \mathcal{F}^k M^{\otimes n}$ for $k = \sum_{i=1}^{n} \lambda \geq l$. □

Proposition 2.37. *Let k be a radically filtered ring over $\mathbb{L}$ with $|\mathbb{L}| > 1$. Let $\mathfrak{a}$ be a filtered k-quiver for which every $\mathfrak{a}(A, A)$ is a free, canonically filtered k-module. Let X be a set, $f : X \longrightarrow \mathrm{Ob}(\mathfrak{a})$ a map, and $(\alpha_x)_x$ with $\alpha_x \in \mathfrak{a}(f(x), f(x))$ a tensor convergent collection of elements. There exists $0 < \lambda$ such that $\alpha_x \in \mathcal{F}^\lambda\mathfrak{a}(f(x), f(x))$ for all $x \in X$.*

Proof. Take any $0 < l$. Since $(\alpha_x)_x$ is tensor convergent, there exists $n \in \mathbb{N}$ for which $\alpha_x^{\otimes n} \in \mathcal{F}^l\mathfrak{a}(f(x), f(x))^{\otimes n}$ for every x. For l' as in (7), by Lemma 2.16, we have $\alpha_x \in \mathcal{F}^{l'}\mathfrak{a}(f(x), f(x))$. □

2.8. **Filtered cocategories.**

Definition 2.38. A *filtered cocategory* $\mathcal{C}$ is a filtered k-quiver with a filtered morphism of quivers

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$$

which is *coassociative*, i.e. Δ satisfies

$$(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta.$$

The morphism Δ is called the *comultiplication*. A *counit* for $\mathcal{C}$ is a filtered morphism $\eta : \mathcal{C} \longrightarrow k\mathrm{Ob}(\mathcal{C})$ with

$$(1_{\mathcal{C}} \otimes \eta)\Delta \cong 1_{\mathcal{C}} \cong (\eta \otimes 1_{\mathcal{C}})\Delta,$$

and in this case $\mathcal{C}$ is called *counital*. An *augmentation* for $\mathcal{C}$ is a filtered retract $\epsilon : k\mathrm{Ob}(\mathcal{C}) \longrightarrow \mathcal{C}$ of the counit, i.e.

$$\eta\epsilon = \mathrm{Id}_{k\mathrm{Ob}(\mathcal{C})}$$

For a filtered cocategory $(\mathcal{C}, \Delta)$, we can iterate the comultiplication. We put

$$\Delta^{(0)} = 1 : \mathcal{C} \longrightarrow \mathcal{C}$$
$$\Delta^{(1)} = \Delta : \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$$
$$\Delta^{(n)} = (1^{\otimes n-1} \otimes \Delta)\Delta^{(n-1)} : \mathcal{C} \longrightarrow \mathcal{C}^{\otimes n+1}$$

Remark 2.39. Let x be an element in $\mathcal{C}$. We will use the notation for the iterated comultiplication $\Delta^{(n)}$ as introduced in [3]:

$$\Delta^{(n)}(x) = \sum_a x_a^1 \otimes \ldots \otimes x_a^{n+1}$$

Definition 2.40. A *filtered morphism of cocategories* $f : (\mathcal{C}, \Delta) \longrightarrow (\mathcal{C}', \Delta')$ is a filtered morphism of k-quivers such that

$$(f \otimes f)\Delta = \Delta' f.$$

If $\mathcal{C}$ and $\mathcal{C}'$ are counital with respective counits η and η', then f is *a filtered morphims of counital cocategories* provided that furthermore

$$k\mathrm{Ob}(f)\eta = \eta' f.$$

Consider two filtered morphisms $f, g : (\mathcal{C}, \Delta) \longrightarrow (\mathcal{C}', \Delta')$. A filtered morphism of k-quivers $d : \mathcal{C} \longrightarrow \mathcal{C}'$ is a *filtered* (f, g)*-coderivation* if and only if

$$\Delta' d = (f \otimes d + d \otimes g)\Delta$$

A *filtered coderivation* of a filtered cocategory $\mathcal{C}$ is a $(1_{\mathcal{C}}, 1_{\mathcal{C}})$-coderivation.

A *filtered dg-cocategory* is a filtered cocategory $(\mathcal{C}, \Delta)$ endowed with a filtered coderivation d such that $d^2 = 0$. In this case, d is called the *codifferential* of $\mathcal{C}$. A *filtered morphism of dg cocategories* $f : (\mathcal{C}, \Delta, d) \longrightarrow (\mathcal{C}', \Delta', d')$ is a filtered morphism of cocategories which satisfies $d' f = f d$.

Remark 2.41. Consider a morphism of cocategories $f : \mathcal{C} \longrightarrow \mathcal{D}$. By definition, we have that

$$\Delta'^{(n)} f = f^{\otimes n} \Delta^{(n)}$$

whereas for an (f, g)-coderivation $d : \mathcal{C} \longrightarrow \mathcal{D}$, we have that

$$\Delta'^{(n)} d = (d \otimes g^{\otimes n} + f \otimes d \otimes g^{\otimes n-1} + \ldots + f^{\otimes n} \otimes d)\Delta^{(n)}$$

Let $(\mathcal{C}, \Delta)$ be a filtered cocategory. We obtain an induced morphism

$$\hat{\Delta} : \hat{\mathcal{C}} \longrightarrow \mathcal{C}\hat{\otimes}\mathcal{C}$$

where $\mathcal{C}\hat{\otimes}\mathcal{C}$ is the completion of $\mathcal{C} \otimes \mathcal{C}$ for its natural filtration. There is a canonical filtered morphism of quivers $\hat{\mathcal{C}} \otimes \hat{\mathcal{C}} \longrightarrow \hat{\mathcal{C}}\hat{\otimes}\hat{\mathcal{C}} \cong \mathcal{C}\hat{\otimes}\mathcal{C}$ which is not an isomorphism in general. In contrast to the dual case of filtered categories, the completion $\hat{\mathcal{C}}$ does not naturally inherit a structure of filtered cocategory. To remedy this, we introduce the following notion:

Definition 2.42. A *formal cocategory* $\mathcal{C}$ is a complete filtered k-quiver with a filtered morphism of quivers

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C}\hat{\otimes}\mathcal{C}$$

which is *formally coassociative*, i.e. Δ satisfies

$$(1\hat{\otimes}\Delta)\Delta = (\Delta\hat{\otimes}1)\Delta.$$

Similarly, the other notions in Definition 2.38 as well as in Definition 2.40 can easily be adapted to the formal setting (with the adjectives *formal* replacing *filtered* in the original definition) simply by working with completed tensor products instead of ordinary filtered tensor products.

Example 2.43. Let $\mathfrak{a}$ be a filtered k-quiver and $B\mathfrak{a}$ its Bar construction as in Definition 2.32. We introduce a filtered counital cocategory structure on $B\mathfrak{a}$. As a consequence, the completion $\hat{B}\mathfrak{a}$ becomes a formal counital cocategory.

The quiver $B\mathfrak{a}$ comes equipped with natural projections $p_n : B\mathfrak{a} \longrightarrow (\Sigma\mathfrak{a})^{\otimes n}$ and injections $i_n : (\Sigma\mathfrak{a})^{\otimes n} \longrightarrow B\mathfrak{a}$. For legibility, we omit the maps i_n from the notations where possible. In particular, for every object $A \in \mathfrak{a}$ we have an element $1_{k,A} \in k\mathrm{Ob}(\mathfrak{a})(A, A) = (\Sigma\mathfrak{a})^{\otimes 0} \subseteq B\mathfrak{a}$. If the object A is clear from the context, we will simply write 1_k.

The quiver $B\mathfrak{a}$ becomes a cocategory with the comultiplication $\Delta : B\mathfrak{a} \longrightarrow B\mathfrak{a} \otimes B\mathfrak{a}$ determined by

$$\Delta(1_k) = 1_k \otimes 1_k$$

$$\Delta(a) = 1_k \otimes a + a \otimes 1_k$$

$$\begin{aligned}\Delta(a_n \otimes \cdots \otimes a_1) =& (a_n \otimes \cdots \otimes a_1) \bigotimes 1_k \\ &+ \sum_{i=1}^{n-1} (a_n \otimes \cdots \otimes a_{i+1}) \bigotimes (a_i \otimes \cdots \otimes a_1) \\ &+ 1_k \bigotimes (a_n \otimes \cdots \otimes a_1)\end{aligned}$$

for $1_k \in (\Sigma\mathfrak{a})^{\otimes 0}$, $a \in \Sigma\mathfrak{a}$, $(a_n \otimes \cdots \otimes a_1) \in (\Sigma\mathfrak{a})^{\otimes n}$. The cocategory $B\mathfrak{a}$ is counital with $p_0 : B\mathfrak{a} \longrightarrow k\mathrm{Ob}(\mathfrak{a})$ as counit.

2.9. **Filtered cA_∞-categories.** Let $\mathfrak{a}$ be a filtered k-quiver. Consider an element

$$m = (m_n)_{n \geq 0} \in \mathbf{C}^2(\mathfrak{a}) \cong [B\mathfrak{a}, \Sigma\mathfrak{a}]^1$$

with the notation of Remark 2.34. Consider for $n \geq 1$ the filtered morphism of quivers $\hat{m}_n : B\mathfrak{a} \longrightarrow (B\mathfrak{a})_n$ which sends $x_1 \otimes \ldots \otimes x_k$ to

$$\sum_{l=1}^{n} (-1)^{|x_1|+\ldots+|x_{l-1}|+l-1} x_1 \otimes \ldots \otimes m_{k-n+1}(x_l, \ldots, x_{l+k-n}) \otimes \ldots \otimes x_k.$$

Lemma 2.44. *There exists a unique filtered coderivation $\bar{m} : B\mathfrak{a} \longrightarrow B\mathfrak{a}$ with $p_1\bar{m} = \hat{m}_1 = m$. The filtered coderivation $\bar{m}$ further satisfies $p_n\bar{m} = \hat{m}_n$. As such, we have $\bar{m} = \sum_n \hat{m}_n$ where, upon evaluation at a fixed element in $B\mathfrak{a}$, the sum becomes finite.*

Remark 2.45. We also obtain an induced formal coderivation $\hat{m} = \hat{\bar{m}} : \hat{B}\mathfrak{a} \longrightarrow \hat{B}\mathfrak{a}$.

Lemma 2.46. *The following are equivalent:*

(1) $m\{m\} = 0$.
(2) $\bar{m}\bar{m} = 0$.
(3) $\hat{m}\hat{m} = 0$.

Definition 2.47. Let $\mathfrak{a}$ be a filtered k-quiver. A *filtered cA_∞-structure* on $\mathfrak{a}$ is an element $m \in \mathbf{C}^2(\mathfrak{a})$ that satisfies the equivalent conditions of Lemma 2.46. In this case $(\mathfrak{a}, m)$ is called a *filtered cA_∞-category.*

Remark 2.48. Explicitely, the cA_∞-relation $m\{m\} = 0$ translates into the following identities for $p \geq 0$:

$$\sum_{j+k+l=p} (-1)^{jk+l} m_{j+l+1}(1^{\otimes j} \otimes m_k \otimes 1^{\otimes l}) = 0. \tag{9}$$

The first few identities are explicitely given by:

$$m_1(m_0) = 0 \tag{10}$$

$$m_1m_1 + m_2(1 \otimes m_0) - m_2(m_0 \otimes 1) = 0 \tag{11}$$

(12)
$$m_1m_2 - m_2(1 \otimes m_1) - m_2(m_1 \otimes 1) + m_3(1 \otimes 1 \otimes m_0) - m_3(1 \otimes m_0 \otimes 1) + m_3(m_0 \otimes 1 \otimes 1) = 0$$

(13)
$$\begin{aligned} &m_1m_3 + m_2(1 \otimes m_2) - m_2(m_2 \otimes 1) + m_3(1 \otimes 1 \otimes m_1) + m_3(1 \otimes m_1 \otimes 1) - m_3(m_1 \otimes 1 \otimes 1) \\ &+m_4(1 \otimes 1 \otimes 1 \otimes m_0) - m_4(1 \otimes 1 \otimes m_0 \otimes 1) + m_4(1 \otimes m_0 \otimes 1 \otimes 1) - m_4(m_0 \otimes 1 \otimes 1 \otimes 1) = 0 \end{aligned}$$

$$\vdots$$

Suppose $\mathfrak{b} \subseteq \mathfrak{a}$ is a full subquiver of a filtered quiver $\mathfrak{a}$, that is $\mathrm{Ob}(\mathfrak{b}) \subseteq \mathrm{Ob}(\mathfrak{a})$ and for $B, B' \in \mathrm{Ob}(\mathfrak{b})$ we have $\mathfrak{b}(B, B') = \mathfrak{a}(B, B')$. Endowing $\mathfrak{b}$ with the natural inherited filtration, there is a canonical projection $\mathbf{C}(\mathfrak{a}) \longrightarrow \mathbf{C}(\mathfrak{b})$ between the Hochschild objects which respects the brace operations. As a consequence, every cA_∞-structure m on $\mathfrak{a}$ restricts to an *inherited cA_∞ structure $m|_\mathfrak{b}$* on $\mathfrak{b}$. With this structure, we will call $\mathfrak{b}$ a *full cA_∞-subcategory* of $\mathfrak{a}$.

Definition 2.49. Consider a filtered cA_∞-structure on $\mathfrak{a}$. The component $m_0 \in \mathbf{C}^2(\mathfrak{a})_0$ is given by

$$(m_0^A)_A \in \prod_{A \in \mathfrak{a}} \mathfrak{a}^2(A, A),$$

and is called the *curvature* of $\mathfrak{a}$. Consider an object $A \in \mathfrak{a}$. The element $m_0^A \in \mathfrak{a}^2(A, A)$ is called the *curvature* of A. The object A is called:

(1) *l-curved* for $l \in \mathbb{L}$ if $m_0^A \in \mathcal{F}^l\mathfrak{a}(A, A)$.
(2) *weakly curved* if it is l-curved for some $l \neq 0$.
(3) *uncurved* if $m_0^A = 0$.

The cA_∞-structure m (or the cA_∞-category $\mathfrak{a}$) is called:

(1) *l-curved* for $l \in \mathbb{L}$ if $m_0 \in \mathcal{F}^l\mathbf{C}(\mathfrak{a})$.
(2) *uniformly weakly curved* if it is l-curved for some $l \neq 0$.
(3) *weakly curved* if every object $A \in \mathfrak{a}$ is weakly curved.
(4) an *A_∞-structure (or A_∞-category)* if $m_0 = 0$.

In the cA_∞-category $\mathfrak{a}$ we distinguish the full cA_∞-subcategories

(1) $\mathfrak{a}_l$ of l-curved objects ($l \in \mathbb{L}$), called the l-curved part of $\mathfrak{a}$.
(2) $\mathfrak{a}_{wc}$ of weakly curved objects, called the weakly curved part of $\mathfrak{a}$.
(3) $\mathfrak{a}_\infty$ of uncurved objects, called the infinity part of $\mathfrak{a}$.

Remark 2.50. Let $\mathfrak{a}$ be a filtered cA_∞-category. The full cA_∞-subcategory

(1) $\mathfrak{a}_l$ is l-curved ($l \in \mathbb{L}$).
(2) $\mathfrak{a}_{wc}$ is weakly curved.
(3) $\mathfrak{a}_\infty$ is an A_∞-category.

Remark 2.51. Suppose we add a top element ∞ to $\mathbb{L}$ in order to obtain $\mathbb{L}^\infty$ as in Example 2.4, and suppose we naturally extend all $\mathbb{L}$-filtered modules M in the definition of the filtered cA_∞-category $\mathfrak{a}$ by $\mathcal{F}^\infty M = 0$. Then $\mathfrak{a}$ naturally becomes an $\mathbb{L}^\infty$-filtered cA_∞-category. An object $A \in \mathfrak{a}$ is ∞-curved if and only if it is uncurved, and the notation $\mathfrak{a}_\infty$ is consistent with the interpretation as full subcategory of ∞-curved objects.

Definition 2.52. A *filtered cdg-category* is a filtered cA_∞-category $(\mathfrak{a}, m)$ with $m_n = 0$ for $n \geq 3$.

Remark 2.53. Cdg-categories are the categorical incarnation of cdg-rings, as introduced by Positselski ([18]).

For a cdg-structure $m = (m_0, m_1, m_2)$, the identities (9) reduce to:

$$m_1(m_0) = 0. \tag{14}$$

$$m_1 m_1 + m_2(1 \otimes m_0) - m_2(m_0 \otimes 1) = 0. \tag{15}$$

$$m_1 m_2 - m_2(1 \otimes m_1) - m_2(m_1 \otimes 1) = 0. \tag{16}$$

$$m_2(1 \otimes m_2) - m_2(m_2 \otimes 1) = 0. \tag{17}$$

Example 2.54. Let k be a commutative ground ring and let $\mathfrak{a}$ be a filtered k-linear category, i.e. the composition

$$m_2 : \mathfrak{a} \otimes \mathfrak{a} \longrightarrow \mathfrak{a}$$

is a morphism of filtered k-quivers satisfying the associativity relation.

For $\mathbb{Z}$-graded $\mathfrak{a}$-objects $M = (M^n)_n$ and $N = (N^n)_n$, let $\mathrm{Hom}(M, N)$ be the $\mathbb{Z}$-graded filtered k-module with $\mathrm{Hom}(M, N)^n = \prod_{i \in \mathbb{Z}} \mathfrak{a}(M^i, N^{i+n})$. A *precomplex* of $\mathfrak{a}$-objects is a $\mathbb{Z}$-graded $\mathfrak{a}$-object M endowed with an element $d_M \in \mathrm{Hom}(M, M)^1$, called the *predifferential*.

We define the filtered quiver $\mathsf{PCom}(\mathfrak{a})$ of precomplexes with

$$\mathsf{PCom}(\mathfrak{a})(M,N) = \mathrm{Hom}(M,N).$$

It follows from a direct calculation that we can endow this quiver with a cdg-structure (m_0, m_1, m_2), where m_2 is the composition of graded $\mathfrak{a}$-morphisms,

$$m_1(f) = m_2(d_N, f) - (-1)^n m_2(f, d_M)$$

for $f \in \mathrm{Hom}(M,N)^n$, and the curvature is given by

$$(m_0)_M = d_M^2 = m_2(d_M, d_M).$$

If $d_M \in \mathcal{F}^l\mathrm{Hom}(M,M)$, then $(m_0)_M \in \mathcal{F}^{2l}\mathrm{Hom}(M,M)$.

We obtain the dg category of *complexes* $\mathsf{Com}(\mathfrak{a}) = \mathsf{PCom}(\mathfrak{a})_\infty$ as the full subcategory of uncurved precomplexes. The predifferential d_M of a complex by definition satisfies $d_M^2 = 0$, and is called the *differential*.

2.10. **Filtered cA_∞-quotients.** Let S be a commutative ground ring and let $\mathfrak{a}$ be an S-linear cA_∞-category.

Definition 2.55. A *cA_∞-ideal* in $\mathfrak{a}$ consists of an S-subquiver $\mathfrak{i} \subseteq \mathfrak{a}$ such that for all $n \geq i \geq 1$, $A_0, \dots, A_n \in \mathfrak{a}$, we have

$$m_n(\mathfrak{a}(A_{n-1}, A_n), \dots, \mathfrak{i}(A_{i-1}, A_i), \dots, \mathfrak{a}(A_0, A_1)) \subseteq \mathfrak{i}(A_0, A_n).$$

Remark 2.56. Note that in Definition 2.55, no condition is imposed upon m_0.

Next we construct a quotient cA_∞-category. Consider the S-quiver $\mathfrak{b} = \mathfrak{a}/\mathfrak{i}$ with $\mathfrak{b}(A,A') = \mathfrak{a}(A,A')/\mathfrak{i}(A,A')$ as S-modules. We denote $\mathfrak{a}(A,A') \longrightarrow \mathfrak{b}(A,A') : a \longmapsto [a]$.

Proposition 2.57. *There exists an S-linear cA_∞-structure $\tilde{m} = (\tilde{m}_n)_{n\geq 0}$ on $\mathfrak{b}$ with*

$$\tilde{m}_0^A = [m_0^A] \in \mathfrak{a}(A,A)/\mathfrak{i}(A,A)$$

and $\tilde{m}_n : \mathfrak{b}(A_n, A_{n-1}) \otimes \cdots \otimes \mathfrak{b}(A_0, A_1) \longrightarrow \mathfrak{b}(A_0, A_n)$ given by

$$\tilde{m}_n([a_n], \dots, [a_1]) = [m_n(a_n, \dots a_1)]$$

for $n \geq 1$.

Proof. The operations $\tilde{m}_n$ are well defined since $\mathfrak{i} \subseteq \mathfrak{a}$ is a cA_∞-ideal, and they obviously satisfy the cA_∞-relation $\tilde{m}\{\tilde{m}\} = 0$. □

Remark 2.58. Suppose k is a filtered S-algebra and $\mathfrak{a}$ a filtered k-linear cA_∞ category. Let $\mathfrak{i} \subseteq \mathfrak{a}$ be a k-subquiver which is a cA_∞-ideal in $\mathfrak{a}$. We can endow $\mathfrak{i}$ with the canonical filtration $\mathcal{F}^l\mathfrak{i} = \mathfrak{i} \cap \mathcal{F}^l\mathfrak{a}$ and $\mathfrak{a}/\mathfrak{i}$ with the filtration $\mathcal{F}^l(\mathfrak{a}/\mathfrak{i}) = \mathcal{F}^l\mathfrak{a}/\mathcal{F}^l\mathfrak{i}$. As such, the quotient $\mathfrak{a}/\mathfrak{i}$ becomes a filtered k-linear cA_∞-category with the structure from Proposition 2.57.

Remark 2.59. If $m_0^A \in \mathfrak{i}(A,A)$ for $A \in \mathfrak{a}$, then $\tilde{m}_0^A = 0 \in \mathfrak{b}(A,A)$ whence A is uncurved for $\tilde{m}$. Consequently, if $m_0^A \in \mathfrak{i}(A,A)$ for all $A \in \mathfrak{a}$, $\tilde{m}$ is an A_∞-structure on $\mathfrak{b}$.

Example 2.60. Let $\mathfrak{a}$ be a filtered k-linear cA_∞-category. For $l \in \mathbb{L}$, the k-subquiver $\mathcal{F}^l\mathfrak{a} \subseteq \mathfrak{a}$ is a cA_∞-ideal. If $\mathfrak{a}$ is l-curved, then the k-linear quotient $\mathfrak{a}/\mathcal{F}^l\mathfrak{a}$ is an A_∞-category.

Remark 2.61. In Example 2.60, if $l \neq 0$ and $\mathcal{F}^l\mathfrak{a} \neq \mathfrak{a}$, the non-trivial A_∞-quotient $\mathfrak{a}/\mathcal{F}^l\mathfrak{a}$ protects $\mathfrak{a}$ against some of the notoriously bad behaviour of cA_∞-categories, as we'll see later on in §3.6.2.

3. Curved functor categories

In this section, we investigate appropriate notions of morphisms between cA_∞-categories and the possibility of defining functor categories. Through the bar construction, a cA_∞-category corresponds to a certain cocategory and it is natural to base the morphisms of cA_∞-categories upon the existence of suitable induced morphisms of cocategories. This leads to the notion of a cA_∞-functor, as in Definition 3.9. This notion is based upon the more primitive notion of qA_∞-functor as in Definition 3.11, for which no structure compatibilities are required. In §3.3, the relation with Positselski's notion of qdg functors is discussed.

One of the standout features of the curved world is that it is in general impossible to view a curved dg algebra as a left or right module over itself. More generally, representable modules over a cA_∞-category can be written down, but fail to define cA_∞-functors. The heart of this section lies in §3.4, where the notion of qA_∞-functor is used in the definition of a functor category $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$ between cA_∞-categories $\mathfrak{a}$ and $\mathfrak{b}$ which is itself cA_∞ (Theorem 3.37). This category has the beautiful property that its uncurved objects are precisely the cA_∞-functors (Proposition 3.27). A variant upon these categories will be used in §4 in the definition and study of module categories.

In §3.5, we introduce the notion of homotopy in order to relax the notion of equivalence between cA_∞-categories to that of cA_∞-homotopy equivalence, and we investigate the relation with the categories $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$.

In §3.6, we discuss the difference between cA_∞-homotopy equivalence in the filtered setting, in comparison with the unfiltered setting where it is known that the corresponding notion trivializes the theory.

3.1. qA_∞**-functors.** Consider filtered quivers $\mathfrak{a}$ and $\mathfrak{b}$. Consider a map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$ and an element $F \in \mathbf{C}^1(\mathfrak{a}, \hat{\mathfrak{b}})_f$, i.e. a collection $F = (F_n)_{n\in\mathbb{N}}$ of filtered elements $F_n : (\Sigma\mathfrak{a})^{\otimes n} \longrightarrow \Sigma\hat{\mathfrak{b}}$ homogeneous of degree 0. We will freely interpret $F = (F_n)_{n\in\mathbb{N}}$ as a collection of filtered elements $F_n : \mathfrak{a}^{\otimes n} \longrightarrow \hat{\mathfrak{b}}$ homogeneous of degree $1 - n$. In particular, we have

$$F_0 = (F_0^A)_A \in \prod_{A\in\mathfrak{a}} \hat{\mathfrak{b}}^1(f(A), f(A)).$$

For $n \geq 1$, consider the morphism of quivers $\hat{F}_n$ given by

$$B\mathfrak{a} \xrightarrow[\Delta^{(n-1)}]{} (B\mathfrak{a})^{\otimes n} \xrightarrow[F^{\otimes n}]{} (\Sigma\hat{\mathfrak{b}})^{\otimes n}$$

and put $\hat{F}_0 = k\mathrm{Ob}(f)p_0$. For $k, n, m \in \mathbb{N}$, let $K^{k,n}$ be the collection of $n+1$-tuples of natural numbers $\underline{k} = (0 = k_0, k_1, \dots, k_n = k)$ with $k_{i-1} \leq k_i$. For $\underline{k} \in K^{k,n}$ and $x = x_1 \otimes \cdots \otimes x_k \in \Sigma\mathfrak{a}^{\otimes k}$, put

$$\hat{F}_{\underline{k}}(x) = (-1)^\alpha F_{k_1}(x_1, \dots, x_{k_1}) \otimes \dots F_{k_i - k_{i-1}}(x_{k_{i-1}+1}, \dots, x_{k_i}) \cdots \otimes F_{k_n - k_{n-1}}(x_{k_{n-1}+1}, \dots, x_{k_n}).$$

Here, if we have $k_{i-1} = k_i$, we insert a factor $F_0(1)$ into the tensor product. We have

$$\hat{F}_n(x) = \sum_{\underline{k} \in K^{k,n}} \hat{F}_{\underline{k}}(x).$$

In the following characterization of morphisms, we use the tensor convergent collections from Definition 2.35.

Lemma 3.1. *The following are equivalent.*

(1) *The element $F_0 = (F_0^A)_A$ is tensor convergent with respect to the filtered quiver $\hat{\mathfrak{b}}$ and the map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\hat{\mathfrak{b}})$.*

(2) *There exists a unique morphism of formal counital cocategories*

$$\hat{F} : \hat{B}\mathfrak{a} \longrightarrow \hat{B}\mathfrak{b}$$

with underlying map f and such that $p_1 \hat{F} = \hat{F}_1 = F$.

In this case, the morphism $\hat{F}$ also satisfies $p_n \hat{F} = \hat{F}_n$ for all $n \geq 0$, and we have $\hat{F} = \sum_{n \in \mathbb{N}} \hat{F}_n$.

Proof. It is easily seen that in order to satisfy (2), we need the expression $\hat{F} = \sum_n \hat{F}_n$ to have a convergent evaluation upon any $x = x_1 \otimes \cdots \otimes x_k \in \Sigma\mathfrak{a}^{\otimes k}$. Let $K_m^{k,n} \subseteq K^{k,n}$ be the subcollection of $\underline{k}$ such that for exactly m of the values $i \in \{1, \dots, n\}$ we have $k_{i-1} = k_i$. Let $K_m^k = \cup_{n \in \mathbb{N}} K_m^{k,n}$. Note that the set K_m^k is finite. We can thus rearrange the expression

$$\sum_{n \in \mathbb{N}} \hat{F}_n(x) = \sum_{m \in \mathbb{N}} (\sum_{n \in \mathbb{N}} \sum_{\underline{k} \in K_m^{k,n}} \hat{F}_{\underline{k}}(x)). \tag{18}$$

Tensor convergence of F_0 is equivalent to the fact that for all $l \in \mathbb{L}$, there exists $m_l \in \mathbb{N}$ with for all $m \geq m_l$, for all $n \in \mathbb{N}$ and for all $\underline{k} \in K_m^{k,n}$, we have

$$\hat{F}_{\underline{k}}(x) \in \mathcal{F}^l \hat{B}\mathfrak{b}.$$

This is in turn equivalent to (18) defining a unique element in $\hat{B}\mathfrak{b}$. □

Remark 3.2. Note that in Lemma 3.1, we have in particular

$$\hat{F}(1_A) = \sum_{m \in \mathbb{N}} (F_0^A)^{\otimes m}.$$

Definition 3.3. Consider filtered quivers $\mathfrak{a}$ and $\mathfrak{b}$. A *qA_∞-functor* from $\mathfrak{a}$ to $\mathfrak{b}$ with underlying map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$ is given by an element $F \in \mathbf{C}^1(\mathfrak{a}, \hat{\mathfrak{b}})_f$ with F_0 tensor convergent.

The qA_∞-functor F is called *strict* if $F_0 = 0$.

We denote the subset of qA_∞-functors with underlying f by

$$\mathbf{C}^1_{q\infty}(\mathfrak{a}, \hat{\mathfrak{b}})_f \subseteq \mathbf{C}^1(\mathfrak{a}, \hat{\mathfrak{b}})_f.$$

The set of all qA_∞ functors from $\mathfrak{a}$ to $\mathfrak{b}$ is denoted by $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$.

Remarks 3.4. (1) Obviously, any element $F \in \mathbf{C}^1(\mathfrak{a}, \hat{\mathfrak{b}})_f$ with $F_0 = 0$ defines a strict qA_∞-functor.

(2) For a strict qA_∞-functor $F \in \mathbf{C}^1(\mathfrak{a}, \hat{\mathfrak{b}})_f$, there is an induced morphism $\bar{F} : B\mathfrak{a} \longrightarrow B\mathfrak{b}$ of counital cocategories, whose completion is $\hat{F}$.

(3) We will use the term qA_∞-functor both for the element $F \in \mathbf{C}^1(\mathfrak{a}, \hat{\mathfrak{b}})_f$ and for the induced morphism of formal counital cocategories $\hat{F} : \hat{B}\mathfrak{a} \longrightarrow \hat{B}\mathfrak{b}$.

Example 3.5. For a quiver $\mathfrak{a}$, consider the map $1 = 1_{\mathrm{Ob}(\mathfrak{a})} : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{a})$. Consider the element $I_\mathfrak{a} \in \mathbf{C}^1(\mathfrak{a}, \hat{\mathfrak{a}})_1$ with

$$(I_\mathfrak{a})_1 = (1_{\mathfrak{a}(A,A)} : \mathfrak{a}(A, A') \longrightarrow \mathfrak{a}(A, A'))_{A,A'}$$

and $(I_\mathfrak{a})_n = 0$ for $n \neq 1$. Then $I_\mathfrak{a}$ is a strict qA_∞-functor, and the corresponding morphism of formal counital cocategories is given by the identity morphism $1_{\hat{B}\mathfrak{a}} : \hat{B}\mathfrak{a} \longrightarrow \hat{B}\mathfrak{a}$.

Proposition 3.6. *Consider filtered quivers* $\mathfrak{a}$, $\mathfrak{b}$ *and* $\mathfrak{c}$ *and maps* $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$, $g : \mathrm{Ob}(\mathfrak{b}) \longrightarrow \mathrm{Ob}(\mathfrak{c})$. *Consider* $F \in \mathbf{C}^1_{q\infty}(\mathfrak{a}, \hat{\mathfrak{b}})_f$ *and* $G \in \mathbf{C}^1_{q\infty}(\mathfrak{b}, \hat{\mathfrak{c}})_g$ *with induced morphisms of formal counital cocategories* $\hat{F} : \hat{B}\mathfrak{a} \longrightarrow \hat{B}\mathfrak{b}$ *and* $\hat{G} : \hat{B}\mathfrak{b} \longrightarrow \hat{B}\mathfrak{c}$. *The composition* $\hat{G}\hat{F} : \hat{B}\mathfrak{a} \longrightarrow \hat{B}\mathfrak{c}$ *is a morphism of formal counital cocategories corresponding to* $G\hat{F} \in \mathbf{C}^1_{q\infty}(\mathfrak{a}, \hat{\mathfrak{c}})_{gf}$ *with*

$$(G\hat{F})(x) = \sum_{n \geq 0} G(\hat{F}_n(x))$$

for $x \in B\mathfrak{a}$.

Definition 3.7. Consider filtered quivers $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. The *composition of* qA_∞*-functors* is given by the operation

$$* : \mathsf{qFun}(\mathfrak{b}, \mathfrak{c}) \times \mathsf{qFun}(\mathfrak{a}, \mathfrak{b}) \longrightarrow \mathsf{qFun}(\mathfrak{a}, \mathfrak{c}) : (G, F) \longmapsto G * F = G\hat{F}.$$

By Proposition 3.6, the operation $*$ is associative, and in accordance with Example 3.5, the morphims $I_\mathfrak{a}$ are identities with respect to $*$. We thus obtain a category of quivers with qA_∞-morphisms, determining the notion of qA_∞-isomorphism as follows:

Definition 3.8. A qA_∞-functor $F \in \mathbf{C}^1_{q\infty}(\mathfrak{a}, \hat{\mathfrak{b}})_f$ is a qA_∞*-isomorphism* if there exists a qA_∞-morphism $G \in \mathbf{C}^1_{q\infty}(\mathfrak{b}, \hat{\mathfrak{a}})_g$, the *inverse isomorphism*, with f and g inverse bijections and $G * F = I_\mathfrak{a}$ and $F * G = I_\mathfrak{b}$.

3.2. cA_∞**-functors.** In order to avoid taking too many completions in our notations, from now on, without loss of generality, *we assume that all original quivers* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}, \dots$ *under consideration are complete with respect to their filtration.*

In order to make a qA_∞-functor into a morphism between filtered cA_∞-categories, we need to impose a compatibility condition with the cA_∞-structures:

Definition 3.9. Consider cA_∞-categories $(\mathfrak{a}, m)$ and $(\mathfrak{b}, m')$. A cA_∞*-functor* from $\mathfrak{a}$ to $\mathfrak{b}$ with underlying map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$ is a qA_∞-functor

$$F \in \mathbf{C}^1_{q\infty}(\mathfrak{a}, \mathfrak{b})_f$$

such that $\hat{F}$ is a complete formal morphism of differential graded cocategories, i.e.

$$\hat{F}\hat{m} = \hat{m}'\hat{F} \tag{19}$$

for $\hat{m}$ and $\hat{m}'$ are as in Remark 2.45 and $\hat{F}$ as in Lemma 3.1.

Remark 3.10. Using the explicit formulations (see Remark 2.45 and Lemma 3.1) in condition (19), we see that $F \in \mathbf{C}^1_{q\infty}(\mathfrak{a}, \mathfrak{b})_f$ defines a cA_∞-functor if and only if it satisfies

(20)
$$\sum_{j+k+l=p} (-1)^{jk+l} F_{j+l+1}(1^{\otimes j} \otimes m_k \otimes 1^{\otimes l}) = \sum_{i_1+\ldots+i_r=p} (-1)^s m'_r(F_{i_1}, \ldots, F_{i_r})$$

where for $p \geq 2$ we have $s = \sum_{2\leq u\leq r} \Big((1-i_u)\sum_{1\leq v\leq u-1} i_v\Big)$, and for $p=1$ we have that $s=1$.

Note that for $p=0$ the right-hand side of (20) is given by
$$m'_0 + m'_1(F_0) + m'_2(F_0, F_0) + \ldots$$
which exists in $\hat{B}\mathfrak{b}$ since m is filtered and $\sum_n F_0^{\otimes n} \in \hat{B}\mathfrak{b}$.

We denote the subset of cA_∞-functors with underlying f by
$$\mathbf{C}^1_{c\infty}(\mathfrak{a}, \mathfrak{b})_f \subseteq \mathbf{C}^1_{q\infty}(\mathfrak{a}, \mathfrak{b})_f.$$
The set of all cA_∞ functors from $\mathfrak{a}$ to $\mathfrak{b}$ is denoted by $\mathsf{cFun}(\mathfrak{a}, \mathfrak{b}) \subseteq \mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$.

By Remark **??**, the composition of qA_∞-functors from Definition 3.7 restricts to a *composition of cA_∞-functors*
$$* : \mathsf{cFun}(\mathfrak{b}, \mathfrak{c}) \times \mathsf{cFun}(\mathfrak{a}, \mathfrak{b}) \longrightarrow \mathsf{cFun}(\mathfrak{a}, \mathfrak{c}) : (G, F) \longmapsto G * F = G\hat{F}.$$

Obviously, we have $I_\mathfrak{a} \in \mathbf{C}^1_{c\infty}(\mathfrak{a}, \hat{\mathfrak{a}})_1$ and we obtain a category of cA_∞-categories with cA_∞-morphisms, determining the notion of *cA_∞-isomorphism* to be a cA_∞-functor which is a qA_∞-isomorphism in the sense of Definition 3.8 whose inverse isomorphism is also a cA_∞-functor.

Definition 3.11. Consider A_∞-categories $(\mathfrak{a}, m)$ and $(\mathfrak{b}, m')$. An *A_∞-functor* (or *A_∞-morphism* or *morphism of A_∞-categories*) from $\mathfrak{a}$ to $\mathfrak{b}$ is a strict cA_∞-functor $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ from $\mathfrak{a}$ to $\mathfrak{b}$.

Next we discuss how the notion of cA_∞-functor is related to related notions in the literature. The conclusions are based upon Propositions 2.36 and 2.37.

Example 3.12. Consider $\mathbb{L} = \mathbb{S} = \{0, \infty\}$. An element $F \in \mathbf{C}^1(\mathfrak{a}, \mathfrak{b})_f$ with $F_0^A = 0$ defines a strict qA_∞-functor, which is cA_∞ as soon as (20) holds. If $\{0\} \subseteq k$ is radical and every $\mathfrak{b}(B, B)$ is a free, canonically filtered k-module, these are the only qA_∞-functors.

Suppose on the other hand that X is a k-module and $\mathfrak{b}$ is the one object quiver $\mathfrak{b} = \mathrm{Hom}(X^n, X^n)$. Then as soon as F_0^A is upper triangular for every A, F is tensor convergent and defines a qA_∞-functor (see [2]).

Example 3.13. Let k be commutative ring with an ideal $I \subseteq k$ and canonically I-adically filtered cA_∞-categories $\mathfrak{a}$ and $\mathfrak{b}$. Note that $\mathfrak{a}$ is weakly curved if and only if it is uniformly weakly curved if and only if it is 1-curved, i.e. $m_0^A \in I\mathfrak{a}(A, A)$ for all $A \in \mathfrak{a}$. In case I is the maximal ideal of a local ring, these are precisely the wcA_∞-categories considered by Positselski in [17, §7]. Consider an element $F \in \mathbf{C}^1(\mathfrak{a}, \mathfrak{b})_f$. Suppose that $F_0^A \in I\mathfrak{a}(f(A), f(A))$ for every $A \in \mathfrak{a}$. Then F defines a qA_∞-functor, which is cA_∞ as soon as (20) holds. If I is a radical ideal, and every $\mathfrak{b}(B, B)$ is free, these are the only qA_∞-functors. In case I is the maximal ideal of a local ring, F is a cA_∞-functor precisely when it is a wcA_∞ functor in the sense of [17, §7].

Example 3.14. Consider $\mathbb{L} = \mathbb{R}^+$, S a reduced commutative ring and $k = S[[\mathbb{R}^+]]$ as in Example 2.20 and canonically k-filtered cA_∞-categories $\mathfrak{a}$ and $\mathfrak{b}$. Consider an element $F \in \mathbf{C}^1(\mathfrak{a}, \mathfrak{b})_f$. Suppose there is a $\lambda \in \mathbb{R}_0^+$ such that $F_0^A \in t^\lambda \mathfrak{a}(f(A), f(A))$ for every $A \in \mathfrak{a}$. Then F defines a qA_∞-functor, which is cA_∞ as soon as (20) holds. If S is reduced, and every $\mathfrak{b}(B,B)$ is free, these are the only qA_∞-functors. In the case of the universal Novikov ring $\mathbb{Q}[\xi][[\mathbb{R}^+]]$ and one object quivers, cA_∞-categories and cA_∞-functors correspond precisely to filtered A_∞-categories and filtered A_∞-functors in the sense of [5, §3.2.2], as used in the context of Fukaya categories.

3.3. **qdg-functors and cdg-functors.** We will now consider cdg-categories, and describe the cdg-analogons of cA_∞-functors, qA_∞-functors and A_∞-functors.

Definition 3.15. Let $(\mathfrak{a}, m)$ and $(\mathfrak{b}, m')$ be cdg-categories. A *cdg-functor* with underlying map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$ is a cA_∞-functor F with $F_n = 0$ for $n \geq 2$.

Proposition 3.16. *Consider cdg-categories* $(\mathfrak{a}, m)$ *and* $(\mathfrak{b}, m')$ *and an element* $F = (F_0, F_1)$ *with*

$$F_0 = (F_A) \in \prod_{A \in \mathfrak{a}} \mathfrak{b}(f(A), f(A))^1$$

and

$$F_1 = (F_{A,A'}) \in \prod_{A,A' \in \mathfrak{a}} \mathrm{Hom}_k^0\Big(\mathfrak{a}(A, A'), \mathfrak{b}(f(A), f(A'))\Big)$$

The element F is a cdg-functor provided the following identities hold:

$$F_1(m_0) = m_0' + m_1'(F_0) + m_2'(F_0, F_0) \tag{21}$$

$$F_1 m_1 = m_1' F_1 - m_2'(F_1 \otimes F_0) + m_2'(F_0 \otimes F_1) \tag{22}$$

$$F_1 m_2 = m_2'(F_1 \otimes F_1) \tag{23}$$

Proof. The identities (21), (22) and (23) are the explicit formulations of the identities (20) in the case of cdg-categories. □

Definition 3.17. [19, §1.4] Let $(\mathfrak{a}, m)$ and $(\mathfrak{b}, m')$ be cdg-categories. A *qdg-functor* with underlying map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$ consists of the same datum $F \in \mathbf{C}^1(\mathfrak{a}, \mathfrak{b})_f$ as a cdg-functor, but from the conditions (21), (22), and (23), condition (21) is omitted.

Remark 3.18. Just like the notion of a qA_∞-functor constitutes a relaxation of the notion of a cA_∞-functor, the notion of a qdg-functor constitutes a relaxation of the notion of a cdg-functor. However, in the qdg case the relaxation is more restrictive as two structure compatibility conditions (22) and (23) remain. It will turn out later on in Lemma 4.2 that the resulting notion is sufficient in order to define represenatble modules over a cdg category, wheras in the general case qA_∞-functors are needed in order to define representable modules over cA_∞-categories.

Definition 3.19. A *dg-functor* between dg categories is a strict cdg functor.

Next, we give an example of a non-strict cdg functor between dg categories.

Example 3.20. Consider the ring k as a one object dg-category, and let the cdg category $\mathsf{PCom}(k)$ and the dg category $\mathsf{Com}(k)$ be as in Example 2.54. We define the functor $F : k \longrightarrow \mathsf{Com}(k)$ by the underlying morphism

$$f : \mathrm{Ob}(k) \longrightarrow \mathrm{Ob}(\mathsf{Com}(k)) : * \mapsto (M, \delta)$$

where M is a complex with a non-zero differential δ (i.e. $\delta \neq 0$, $\delta^2 = 0$), and components F_0, F_1 given by

$$\begin{aligned} F_0 &: k \longrightarrow \mathsf{Com}^1(M, M) : 1_k \mapsto -\delta \\ F_1 &: k \longrightarrow \mathsf{Com}(M, M) : 1_k \mapsto Id_M. \end{aligned}$$

This is indeed defines a cdg-functor F, since we have that

$$\begin{aligned} m_{0,M}^{\mathsf{Com}} + m_1(F_0) + m_2(F_0, F_0) &= 0 + \delta F_0 - (-1)F_0\delta + F_0F_0 \\ &= 0 = m_0^k \\ m_1(F_1(1_k)) + m_2(F_0, F_1(1_k) - m_2(F_1(1_k), F_0) &= \delta Id_M - Id_M\delta + (-\delta)Id_M + Id_M\delta \\ &= 0 = F_1(m_1(1_k)) \\ F_1(m_2(1_k, 1_k)) &= F_1(1_k) \\ &= Id_M = m_2(F_1(1_k), F_1(1_k)). \end{aligned}$$

By construction, the cdg-functor F is non-strict. Note that the example can be repeated without effort for $\mathsf{PCom}(k)$ instead of $\mathsf{Com}(k)$.

3.4. **The functor categories qFun and cFun.** In order to consider the category $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$ of qA_∞-functors from $\mathfrak{a}$ to $\mathfrak{b}$, we need a notion of prenatural transformations between qA_∞-functors. This notion will be inspired upon the definitions and constructions from [3, §7], modified to meet the qA_∞ framework.

Let $F, G : \mathfrak{a} \longrightarrow \mathfrak{b}$ be qA_∞-functors with underlying morphism respectively f and g. Consider for all $A, B \in \mathfrak{a}$ collections of filtered morphisms, which are homogeneous of degree t

$$\eta_k(A, B) : B_k\mathfrak{a}(A, B) \longrightarrow \Sigma\mathfrak{b}(f(A), g(B))$$

with $k \geq 1$ for $A \neq B$ and $k \geq 0$ for $A = B$.

From now we simply write $\eta_k : B_k\mathfrak{a} \longrightarrow \Sigma\mathfrak{b}$ as we assume the objects to be clear from the context.

Proposition 3.21. *(see* [3, Lemma 7.45]*). For each family* $(\eta_k : B_k\mathfrak{a} \longrightarrow \Sigma\mathfrak{b})_k$ *as above, there is a unique* $(\hat{G}, \hat{F})$*-coderivation* $\hat{\eta} : \hat{B}\mathfrak{a} \longrightarrow \hat{B}\mathfrak{b}$ *such that*

$$p_1\hat{\eta}|_{B_k\mathfrak{a}} = \eta_k.$$

Proof. Using the notation of Remark 2.39, it is clear that the required morphism $\hat{\eta}$ is given by

$$\hat{\eta}(x) = \sum_a (-1)^{|\eta||x_a^1|}\hat{G}(x_a^1) \otimes \eta(x_a^2) \otimes \hat{F}(x_a^3)$$

where $\eta(x_a^2) = \eta_k(x_a^2)$ when $x_a^2 \in B_k\mathfrak{a}$. □

Remark 3.22. Note that the expression $\hat{\eta}$ is well-defined by the definition of a qA_∞-functor.

Definition 3.23. We call a family $\eta = (\eta_k)_k$ as in Proposition 3.21, or the associated $(\hat{G}, \hat{F})$-coderivation $\hat{\eta}$ a *pre-natural transformation from F to G.*

Notation 3.24. At this point, we would like to comment upon the notations. Firstly, we will always denote

(1) the multiplicative structure on a quiver by the letter m.
(2) functors by capital letters F, G, ...
(3) pre-natural transformations by greek letters η, κ, ...

Secondly, the notation $\hat{(-)}$ is used to indicate the morphism on the level of the completed bar-constructions associated to the argument. We have encountered three such associated morphisms, namely $\hat{m}$, $\hat{F}$ and $\hat{\eta}$. The first one is associated to a multiplicative structure m on a quiver (see Remark 2.45), the second one is associated to a functor F (see Proposition 3.1) and the last one is associated to a pre-natural transformation η (see Proposition 3.21). Note that in each case, the construction of $\hat{(-)}$ is different, yet we use the same notation.

In the light of future calculations, we now look at some of the interactions between the different $\hat{(-)}$ constructions.

Proposition 3.25. *Consider cA_∞-categories $(\mathfrak{a}, m)$ and $(\mathfrak{b}, m')$, qA_∞-functors $F, G : \mathfrak{a} \longrightarrow \mathfrak{b}$ and a pre-natural transformation $\eta : F \longrightarrow G$. We then have that*

(1) $\hat{m}'\hat{F} = \hat{F} \otimes m'(\hat{F}) \otimes \hat{F}$
(2) $\hat{F}\hat{m} = \hat{F} \otimes F(\hat{m}) \otimes \hat{F}$
(3) $\hat{m}'\hat{\eta} = \hat{m}'\hat{G} \otimes \eta \otimes \hat{F} + \hat{G} \otimes m'(\hat{\eta}) \otimes \hat{F} + \hat{G} \otimes \eta \otimes \hat{m}'\hat{F}$
(4) $\hat{\eta}\hat{m} = \hat{G}\hat{m} \otimes \eta \otimes \hat{F} + \hat{G} \otimes \eta(\hat{m}) \otimes \hat{F} + \hat{G} \otimes \eta \otimes \hat{F}\hat{m}$

Proof. Consider $x = x_1 \otimes \ldots \otimes x_r \in B_r\mathfrak{a}$

(1)

$$\begin{aligned}
\hat{m}'\hat{F}(x) &= \hat{m}'(\sum_n \hat{F}_n(x)) \\
&= \hat{m}'(\sum_n \sum_a F(x_a^1) \otimes \ldots \otimes F(x_a^n)) \\
&= \sum_n \sum_{j+k+l=n} (1^{\otimes j} \otimes m'_k \otimes 1^{\otimes l})(\sum_a F(x_a^1) \otimes \ldots \otimes F(x_a^n)) \\
&= \sum_n (\sum_{j+k+l=n} F(x_a^1) \otimes \ldots \otimes F(x_a^j) \otimes m'_k(F(x_a^{j+1}) \otimes \ldots \otimes F(x_a^{j+k})) \otimes F(x_a^{j+k+1}) \otimes \ldots \otimes F(x_a^n)) \\
&= \hat{F} \otimes m'(\hat{F}) \otimes \hat{F}
\end{aligned}$$

(2)

$$\begin{aligned}
\hat{F}\hat{m}(x) &= \hat{F}\big(\sum_{j+k+l=n} x_1 \otimes \ldots \otimes x_j \otimes m_k(x_{j+1} \otimes \ldots \otimes x_{j+k}) \otimes x_{j+k+1} \otimes \ldots \otimes x_r\big) \\
&= \sum_n \sum_a F(y_a^1) \otimes \ldots \otimes F(x_u \otimes \ldots \otimes m_q(x_{v+1} \otimes \ldots \otimes x_{v+q}) \otimes \ldots \otimes x_w) \otimes \ldots \otimes F(y_a^n) \\
&= \hat{F} \otimes F(\hat{m}) \otimes \hat{F}
\end{aligned}$$

where

$$\Delta^{(n-1)}(x_1 \otimes \ldots \otimes x_j \otimes m_k(x_{j+1} \otimes \ldots \otimes x_{j+k}) \otimes x_{j+k+1} \otimes \ldots \otimes x_r) = \sum_a y_a^1 \otimes \ldots \otimes y_a^n$$

and $x_u \otimes \ldots \otimes m_q(x_{v+1} \otimes \ldots \otimes x_{v+q}) \otimes \ldots \otimes x_w = y_a^j$ for a certain j.

(3) Analogous to (1).

(4) Analogous to (2).

□

3.4.1. *The cA_∞-category* $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$. Let $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$ be the k-quiver consisting of the qA_∞-functors $\mathfrak{a} \longrightarrow \mathfrak{b}$ as objects, and morphisms-sets given by the $\mathbb{Z}$-graded modules of pre-natural transformations. Consider F, G two qA_∞-functors in $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$. In order to simplify the notation, we denote the set of morphisms $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})(F, G)$ by $\mathsf{qFun}(F, G)$ when the categories $\mathfrak{a}$ and $\mathfrak{b}$ are clear from the context.

The k-quiver $\mathsf{qFun}(F, G)$ is endowed with an induced filtration over $\mathbb{L}$, namely for $l \in \mathbb{L}$ the component $\mathcal{F}^l\mathsf{qFun}(F, G)$ is defined as

$$\mathcal{F}^l\mathsf{qFun}(F, G) = \{\eta \in \mathsf{qFun}(F, G) \mid \forall p \in \mathbb{L}, \forall k \in \mathbb{N} : \ \eta_k(\mathcal{F}^p\mathfrak{a}^{\otimes k}) \subset \mathcal{F}^{p+l}\mathfrak{b}\}$$

We will now introduce multiplications $(\mathfrak{M}_k)_k$ on this quiver, that will endow it with the structure of a cA_∞-category.

Definition 3.26. The *curvature* $\mathfrak{M}_0^F$ of a qA_∞-functor $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ is defined by

$$\mathfrak{M}_0^F = p_1(\hat{m}'\hat{F} - \hat{F}\hat{m}).$$

Hence, the curvature somehow measures how close the functor is to being a cA_∞-functor. In particular, we have:

Proposition 3.27. *A qA_∞-functor F is a cA_∞-functor if and only if*

$$\mathfrak{M}_0^F = 0.$$

Lemma 3.28. *Let $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ be a qA_∞-functor. Then the expression $\hat{m}'\hat{F} - \hat{F}\hat{m}$ is an $(\hat{F}, \hat{F})$-coderivation.*

Proof. We have:

$$\begin{aligned}\Delta(\hat{m}'\hat{F} - \hat{F}\hat{m}) &= (\hat{m}' \otimes 1 + 1 \otimes \hat{m}')\Delta\hat{F} - (\hat{F} \otimes \hat{F})\Delta\hat{m} \\ &= (\hat{m}' \otimes 1 + 1 \otimes \hat{m}')(\hat{F} \otimes \hat{F})\Delta - (\hat{F} \otimes \hat{F})(\hat{m} \otimes 1 + 1 \otimes \hat{m})\Delta \\ &= \Big(\hat{F} \otimes (\hat{m}'\hat{F} - \hat{F}\hat{m}) + (\hat{m}'\hat{F} - \hat{F}\hat{m}) \otimes \hat{F}\Big)\Delta\end{aligned}$$

□

Remark 3.29. By Proposition 3.21, the previous lemma implies that

$$\widehat{\mathfrak{M}_0^F} = \hat{m}'\hat{F} - \hat{F}\hat{m}$$

Definition 3.30. The *multiplication* $\mathfrak{M}_1 : \mathsf{qFun}(F, G) \longrightarrow \mathsf{qFun}(F, G)$ is defined by

$$\mathfrak{M}_1(\eta) = p_1\delta\eta$$

where $\delta\eta = \hat{m}'\hat{\eta} + (-1)^{|\eta|}\hat{\eta}\hat{m}$.

Proposition 3.31. *Let η be an element of $\mathsf{qFun}(F,G)$. We then have that*

$$\Delta\delta\eta - \left(\hat{G}\otimes\delta\eta + \delta\eta\otimes\hat{F}\right)\Delta = \left(\mathfrak{M}_0^G\otimes\hat{\eta} + \hat{\eta}\otimes\mathfrak{M}_0^F\right)\Delta$$

Proof.

$$\begin{aligned}\Delta\delta\eta &= \Delta\big(\hat{m}'\hat{\eta} + (-1)^{|\eta|}\hat{\eta}\hat{m}\big)\\ &= (\hat{m}'\otimes 1 + 1\otimes\hat{m}')\Delta\hat{\eta} + (-1)^{|\eta|}\Delta\hat{\eta}\hat{m}\\ &= (\hat{m}'\otimes 1 + 1\otimes\hat{m}')(\hat{G}\otimes\hat{\eta} + \hat{\eta}\otimes\hat{F})\Delta\\ &\quad + (-1)^{|\eta|}(\hat{G}\otimes\hat{\eta} + \hat{\eta}\otimes\hat{F})\Delta\hat{m}\\ &= (\hat{m}'\otimes 1 + 1\otimes\hat{m}')(\hat{G}\otimes\hat{\eta} + \hat{\eta}\otimes\hat{F})\Delta\\ &\quad + (-1)^{|\eta|}(\hat{G}\otimes\hat{\eta} + \hat{\eta}\otimes\hat{F})(\hat{m}\otimes 1 + 1\otimes\hat{m})\Delta\\ &= \Big(\hat{G}\otimes(\hat{m}'\hat{\eta} + (-1)^{|\eta|}\hat{\eta}\hat{m}) + (\hat{m}'\hat{\eta} + (-1)^{|\eta|}\hat{\eta}\hat{m})\otimes\hat{F}\Big)\Delta\\ &\quad + \left(\mathfrak{M}_0^G\otimes\hat{\eta} + \hat{\eta}\otimes\mathfrak{M}_0^F\right)\Delta\\ &= \left(\hat{G}\otimes\delta\eta + \delta\eta\otimes\hat{F}\right)\Delta + \left(\mathfrak{M}_0^G\otimes\hat{\eta} + \hat{\eta}\otimes\mathfrak{M}_0^F\right)\Delta\end{aligned}$$

□

Remark 3.32. The previous proposition shows that the expression $\delta\eta$ is in general not a $(\hat{G},\hat{F})$-coderivation. This is why we have to define the multiplication $\mathfrak{M}_1$ be means of the projection $p_1\delta\eta$, and thus $\mathfrak{M}_1$ is a kind of "coderivationification" of $\delta\eta$.

Since the obstruction to being a coderivation is expressed by means of the curvature elements, by Proposition 3.21 the associated coderivation $\widehat{\delta\eta}$ is not given by $\hat{m}'\hat{\eta} + (-1)^{|\eta|}\hat{\eta}\hat{m}$, unless both F and G are cA_∞-functors.

Definition 3.33. We call a pre-natural transformation η a *natural transformation* if and only if it is $\mathfrak{M}_1$-closed, i.e. $\mathfrak{M}_1(\eta) = 0$.

Remark 3.34. Writing out the condition that $\mathfrak{M}_1(\eta) = 0$, we see that this is equivalent to the identity

$$\sum_{i_1+i_2+i_3=p} (-1)^{\xi} m'_{i_1+i_3+1}((\hat{G})_{i_1}, \eta_{i_2}, (\hat{F})_{i_3}) = \sum_{j+k+l=p} (-1)^{jk+l}\eta_{j+l+1}(1^{\otimes j}\otimes m_k\otimes 1^{\otimes l})$$

where $(\hat{F})_{i_3}$ is the restriction of $\hat{F}$ to $B_{i_3}\mathfrak{a}$, and ξ is defined by the rule explained in the proof of Proposition 3.21.

Finally, we define the higher multiplications $\mathfrak{M}_k : \mathsf{qFun}^{\otimes k}(F,G) \longrightarrow \mathsf{qFun}(F,G)$ for $k \geq 2$. To simplify the notations even further, we write $m(x)$ instead of $m_k(x)$ for $x \in B_k(\mathfrak{a})$. Let F_i, $i = 0,\dots,k$ be qA_∞-functors from $\mathfrak{a}$ to $\mathfrak{b}$ with $F_0 = F$ and $F_k = G$. Further, consider $\eta_i \in \mathsf{qFun}(F_{i-1},F_i)$, $i = 1,\dots,k$ pre-natural transformations, and $x \in B\mathfrak{a}$. We define a pre-natural transformation $T = (T_l : B_l\mathfrak{a} \to \Sigma\mathfrak{b})_l$ given by

$$T_l(x) = -\sum_a (-1)^{\epsilon_a} m'\Big(\hat{F}_k(x_a^1), \eta_k(x_a^2), \dots, \eta_1(x_a^{2k}), \hat{F}_0(x_a^{2k+1})\Big)$$

where $\epsilon_a = \sum_{j=1}^k\sum_{i=1}^{2j-1}|\eta_j||x_a^i|$.

Remark 3.35. This expression $T_l(x)$ is well-defined, since we assumed our cA_∞-categories (thus in particular $\mathfrak{b}$) to be complete.

Definition 3.36. We define the *multiplication* $\mathfrak{M}_k(\eta_k, \dots, \eta_1)$, for $k \geq 2$ by

$$\mathfrak{M}_k(\eta_k, \dots, \eta_1) = T.$$

Theorem 3.37. *The quiver $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$ endowed with the multiplications*

$$\mathfrak{M} = (\mathfrak{M}_k)_{k \geq 0} \in \mathbf{C}^2\left(\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})\right)$$

defined as above, has the structure of a cA_∞-category (i.e. $\hat{\mathfrak{M}}\hat{\mathfrak{M}} = 0$).

Proof. The fact that $\mathfrak{M}$ defines a cA_∞-structure, is expressed by the identity

$$\sum_{j+k+l=p} (-1)^{jk+l} \mathfrak{M}_{j+l+1}(1^{\otimes j} \otimes \mathfrak{M}_k \otimes 1^{\otimes l}) = 0$$

We start with $p = 0$. Since $|\mathfrak{M}_0^F| = 2$, we have

$$\begin{aligned}\mathfrak{M}_1(\mathfrak{M}_0^F) &= p_1\Big(\hat{m}'(\hat{m}'\hat{F} - \hat{F}\hat{m}) + (-1)^{|\mathfrak{M}_0^F|}(\hat{m}'\hat{F} - \hat{F}\hat{m})\hat{m}\Big) \\ &= 0\end{aligned}$$

Take $p = 1$, and consider a pre-natural transformation η. Using the equalities from Proposition 3.25, we have

$$\begin{aligned}
\mathfrak{M}_1\mathfrak{M}_1(\eta) &= \mathfrak{M}_1\left(p_1\left[\hat{m}'(\sum \hat{G} \otimes \eta \otimes \hat{F}) + (-1)^{|\eta|}(\sum \hat{G} \otimes \eta \otimes \hat{F})\hat{m}\right]\right) \\
&= p_1\Big[\hat{m}'(\sum \hat{G} \otimes \Big[p_1\left[\hat{m}'(\sum \hat{G} \otimes \eta \otimes \hat{F}) + (-1)^{|\eta|}(\sum \hat{G} \otimes \eta \otimes \hat{F})\hat{m}\right]\Big] \otimes \hat{F}) \\
&\quad + (-1)^{|\eta|+1}(\sum \hat{G} \otimes \Big[p_1\left[\hat{m}'(\sum \hat{G} \otimes \eta \otimes \hat{F}) + (-1)^{|\eta|}(\sum \hat{G} \otimes \eta \otimes \hat{F})\hat{m}\right]\Big] \otimes \hat{F})\hat{m}\Big] \\
&= m'\Big(\sum \hat{G} \otimes m'(\sum \hat{G} \otimes \eta \otimes \hat{F}) \otimes \hat{F}\Big) + (-1)^{|\eta|} m'(\sum \hat{G} \otimes \eta\hat{m} \otimes \hat{F}) \\
&\quad + (-1)^{|\eta|+1} m'(\sum \hat{G} \otimes \eta \otimes \hat{F})\hat{m} - \eta \underbrace{\hat{m}\hat{m}}_{=0} \\
&= m'\Big(\sum \hat{G} \otimes m'(\sum \hat{G} \otimes \eta \otimes \hat{F}) \otimes \hat{F}\Big) + (-1)^{|\eta|+1}\Big(m'(-\sum \hat{G}\hat{m} \otimes \eta \otimes \hat{F} + \sum \hat{G} \otimes \eta \otimes \hat{F}\hat{m})\Big) \\
&= \underbrace{m'\hat{m}'}_{=0}\Big(\sum \hat{G} \otimes (\sum \hat{G} \otimes \eta \otimes \hat{F}) \otimes \hat{F}\Big) + m'\Big(\sum \hat{G} \otimes m'(\hat{G}) \otimes \hat{G} \otimes \eta \otimes \hat{F}\Big) \\
&\quad - m'\Big(\sum \hat{G} \otimes \eta \otimes \hat{F} \otimes m'(\hat{F}) \otimes \hat{F}\Big) \\
&\quad + (-1)^{|\eta|+1}\Big(m'(-\sum \hat{G} \otimes G\hat{m} \otimes \hat{G} \otimes \eta \otimes \hat{F}) + m'(\sum \hat{G} \otimes \eta \otimes \hat{F} \otimes F\hat{m} \otimes \hat{F})\Big) \\
&= \sum m'(\hat{G} \otimes \eta \otimes \hat{F} \otimes p_1(\hat{F}\hat{m} - \hat{m}'\hat{F}) \otimes \hat{F}) - \sum m'(\hat{G} \otimes p_1(\hat{G}\hat{m} - \hat{m}'\hat{G}) \otimes \hat{G} \otimes \eta \otimes \hat{F}) \\
&= -\mathfrak{M}_2(\eta, \mathfrak{M}_0^F) + \mathfrak{M}_2(\mathfrak{M}_0^G, \eta)
\end{aligned}$$

Here, the signs coming from the expressions $\hat{m}$, $\hat{\eta}$ and $\widehat{\mathfrak{M}_1(\eta)}$ have been used but suppressed from the notation. An explicit calculation of the signs is given in the last part of the proof of [3, Theorem-Definition 7.55].

The rest of the identities (with $p \geq 2$) can be proven analogously to the proof of [3, Theorem-Definition 7.55], keeping in mind that the different expression for $\mathfrak{M}_1$ will yield the appropriate apparitions of $\mathfrak{M}_0$, as it did in the case of $p = 1$. □

Remark 3.38. Consider the functor category $\mathsf{qFun}(\mathfrak{a},\mathfrak{b})$, with $\mathfrak{a}$ a cA_∞-category, and $\mathfrak{b}$ a cdg-category. It is a direct consequence of the definition of the cA_∞-structure $(\mathfrak{M}_k)_k$ that $\mathsf{qFun}(\mathfrak{a},\mathfrak{b})$ is a cdg-category whenever $\mathfrak{b}$ is a cdg-category.

3.4.2. *The A_∞-category* $\mathsf{cFun}(\mathfrak{a},\mathfrak{b})$*.* We now consider the relation between the different functor categories qFun, cFun and Fun of qA_∞-functors, resp. cA_∞-functors and A_∞-functors.

Definition 3.39. Consider two cA_∞-categories $\mathfrak{a}$ and $\mathfrak{b}$. The *cA_∞-functor category* $\mathsf{cFun}(\mathfrak{a},\mathfrak{b})$ is the full cA_∞-subcategory

$$\mathsf{cFun}(\mathfrak{a},\mathfrak{b}) = \big(\mathsf{qFun}(\mathfrak{a},\mathfrak{b})\big)_\infty$$

consisting of the the cA_∞-functors from $\mathfrak{a}$ to $\mathfrak{b}$.

Remark 3.40. By definition, $\mathsf{cFun}(\mathfrak{a},\mathfrak{b})$ is an A_∞-category. It is the cA_∞-incarnation of the classical A_∞-structure of the A_∞-functor category ([3, Theorem-Definition 7.55]).

By Proposition 3.31, the differential on $\mathsf{cFun}(\mathfrak{a},\mathfrak{b})$ is given by

$$\widehat{\mathfrak{M}_1(\eta)} = \hat{m}'\hat{\eta} + (-1)^{|\eta|}\,\hat{\eta}\hat{m}.$$

Definition 3.41. Consider two A_∞-categories $\mathfrak{a}$ and $\mathfrak{b}$. The category $\mathsf{Fun}(\mathfrak{a},\mathfrak{b})$ is the full A_∞-subcategory of $\mathsf{cFun}(\mathfrak{a},\mathfrak{b})$, obtained by restricting the objects to the A_∞-functors from $\mathfrak{a}$ to $\mathfrak{b}$, i.e. the strict cA_∞-functors.

We will end this section by giving a description of the Hochschild complex $\mathbf{C}(\mathfrak{a})$ of a cA_∞-category $\mathfrak{a}$ by means of the functor category $\mathsf{Fun}(\mathfrak{a},\mathfrak{a})$.

Proposition 3.42. *The Hochschild complex of a filtered cA_∞-category $\mathfrak{a}$ is described by the complex*

$$\mathsf{Fun}(\mathfrak{a},\mathfrak{a})(Id_{\mathfrak{a}}, Id_{\mathfrak{a}})$$

Proof. We know by definition that a pre-natural transformation $\eta : \mathrm{Id}_{\mathfrak{a}} \longrightarrow \mathrm{Id}_{\mathfrak{a}}$ is given by homogeneous components of degree n

$$\begin{aligned}
\eta_0 &\in \prod_{A\in\mathfrak{a}} \Sigma\mathfrak{a}(A,A)\\
\eta_1 &\in \prod_{A,B\in\mathfrak{a}} \mathrm{Hom}(\Sigma\mathfrak{a}(A,B),\Sigma\mathfrak{a}(A,B))\\
\eta_2 &\in \prod_{A,B,C\in\mathfrak{a}} \mathrm{Hom}(\Sigma\mathfrak{a}(B,C)\otimes\Sigma\mathfrak{a}(A,B),\Sigma\mathfrak{a}(A,C))\\
&\vdots
\end{aligned}$$

As such, it is clear that $\eta \in \mathbf{C}^n(\mathfrak{a})$. We also have that

$$\begin{aligned}
\mathfrak{M}_1(\eta) &= m(\hat{\eta}) + (-1)^{|\eta|}\eta(\hat{m})\\
&= m(\sum \hat{\mathrm{Id}}_{\mathfrak{a}}\otimes\eta\otimes\hat{\mathrm{Id}}_{\mathfrak{a}}) + (-1)^{|\eta|}\eta(\hat{m})\\
&= m\{\eta\} + (-1)^{|\eta|}\eta\{m\}
\end{aligned}$$

We have thus proven that the Hochschild complex $\mathbf{C}(\mathfrak{a})$ can equivalently be described as $\mathsf{Func}(\mathfrak{a},\mathfrak{a})(\mathrm{Id}_{\mathfrak{a}},\mathrm{Id}_{\mathfrak{a}})$. ☐

Remark 3.43. Take $F \in \mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$. The components of its curvature element are given by

$$\begin{aligned}
\mathfrak{M}_0^F|_0 &= m_0^{\mathfrak{b}} + m_1(F_0) + m_2(F_0, F_0) + \ldots - F_1(m_0^{\mathfrak{a}}) \\
\mathfrak{M}_1^F|_0 &= m_1(F_1) + m_2(F_1, F_0) + m_2(F_0, F_1) + \ldots - F_1(m_1) - F_2(-, m_0^{\mathfrak{a}}) + F_2(m_0^{\mathfrak{a}}, -) \\
\mathfrak{M}_2^F|_0 &= m_1(F_2) + m_2(F_2, F_0) + m_2(F_0, F_2) + m_2(F_1, F_1) + \ldots \\
&\quad - F_1(m_2) + F_2(-, m_1) + F_2(m_1, -) - F_3(m_0^{\mathfrak{a}}, -, -) + F_3(-, m_0^{\mathfrak{a}}, -) - F_3(-, -, m_0^{\mathfrak{a}}) \\
&\ \vdots
\end{aligned}$$

Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are (uniformly) weakly curved. By definition of the multiplicative structures $m^{\mathfrak{a}}$ and $m^{\mathfrak{b}}$, and of the functor F, we know that $\mathfrak{M}_0^F|_0(1) \in \mathcal{F}^l\mathfrak{b}$ for some $l > 0$. However, we cannot conclude for any $l > 0$ that for all $p \in \mathbb{L}$ we have that

$$\mathfrak{M}_0^F|_k(\mathcal{F}^p\mathfrak{a}^{\otimes k}) \subset \mathcal{F}^{l+p}\mathfrak{b}.$$

As such, the natural filtration on $\mathsf{qFun}(\mathfrak{a}, \mathfrak{b})$ need not be weakly curved.

3.5. ***cA_∞*-equivalences.** Based upon the notions of cA_∞-categories, cA_∞-functors and natural transformations between them, we obtain a 2-category of cA_∞-categories. By introducing the notion of homotopy, we arrive at the more relaxed notions of homotopic rather than isomorphic cA_∞-functors and homotopy equivalent rather than equivalent cA_∞-categories respectively.

In this section, we extend work by Fukaya from [3] on A_∞-homotopy equivalences and functor categories to our setup of filtered cA_∞-categories and categories of cA_∞-functors. The next two definitions generalize [3, Definition 8.1] and [3, Definition 8.5].

Definition 3.44. Consider cA_∞-categories $\mathfrak{a}$ and $\mathfrak{b}$. Two cA_∞-functors $F, G : \mathfrak{a} \longrightarrow \mathfrak{b}$ are *homotopic* to each other if and only if there are natural transformation $\eta : F \longrightarrow G$, $\mu : G \longrightarrow F$, and pre-natural transformations $\kappa : G \longrightarrow G$ and $\kappa' : F \longrightarrow F$ such that

$$\begin{aligned}
\mathfrak{M}_2(\eta, \mu) - \mathrm{Id}_G &= \mathfrak{M}_1(\kappa) \\
\mathfrak{M}_2(\mu, \eta) - \mathrm{Id}_F &= \mathfrak{M}_1(\kappa')
\end{aligned}$$

where the natural transformation $\mathrm{Id}_F : F \longrightarrow F$ is defined by $(\mathrm{Id}_F)_0 = \mathrm{Id}_{F(X)}$, and $(\mathrm{Id}_F)_n = 0$ for all $n \geq 1$.

Definition 3.45. A functor $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ between cA_∞-categories is a *cA_∞-homotopy equivalence* if and only if there exists a cA_∞-functor $G : \mathfrak{b} \longrightarrow \mathfrak{a}$ such that $F * G$ is homotopic to $\mathrm{Id}_{\mathfrak{b}}$ and $G * F$ is homotopic to $\mathrm{Id}_{\mathfrak{a}}$.

A functor $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ between A_∞-categories is an *A_∞-homotopy equivalence* if and only if there exists an A_∞-functor $G : \mathfrak{b} \longrightarrow \mathfrak{a}$ such that $F * G$ is homotopic to $\mathrm{Id}_{\mathfrak{b}}$ and $G * F$ is homotopic to $\mathrm{Id}_{\mathfrak{a}}$.

Remark 3.46. The operation $*$ is the composition of qA_∞-functors, as defined in Definition 3.7.

In the A_∞-setup, we have the following important result:

Proposition 3.47. [15, Theorem 8.8] *An A_∞-functor $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ between A_∞-categories is an A_∞-homotopy equivalence if and only if the morphisms*

$$F_1 : (\mathfrak{a}(A, A'), m_1^{\mathfrak{a}}) \longrightarrow (\mathfrak{b}(f(A), f(A')), m_1^{\mathfrak{b}})$$

are homotopy equivalences of chain complexes and the induced functor $H^0F : H^0\mathfrak{a} \longrightarrow H^0\mathfrak{b}$ is essentially surjective.

Remark 3.48. In the case where k is a field, it is well known that homotopy equivalences and quasi-isomorphisms coincide. As such, over a field k, an A_∞-functor F is an A_∞-homotopy equivalence if and only if F_1 is a quasi-isomorphism, and F is essentially surjective in H^0. Over an arbitrary ring, we call this latter notion a *quasi-equivalence.*

The next two propositions extend [3, Prop 8.41].

Proposition 3.49. *Consider cA_∞-categories $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. A cA_∞-functor $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ induces a strict cA_∞-functor $F^* : \mathsf{qFun}(\mathfrak{b}, \mathfrak{c}) \longrightarrow \mathsf{qFun}(\mathfrak{a}, \mathfrak{c})$ with underlying morphism $f^*(G) = G * F$, where $*$ is the composition of qA_∞-functors (see Definition 3.7).*

Proof. Let $\eta_i \in \mathsf{qFun}(\mathfrak{b}, \mathfrak{c})(G_{i-1}, G_i)$ be a pre-natural transformation of degree t_i. We define F^* by

$$(F^*)_1(\eta_1)(x) = \eta_1(\hat{F}(x))$$
$$(F^*)_k(\eta_k, \dots, \eta_1)(x) = 0 \quad \forall k \geq 2$$

where $x \in B\mathfrak{a}$.

This is indeed a cA_∞-functor, since we have for $p = 0$

$$\begin{aligned} (F^*)_1(\mathfrak{M}_{0,G}^{\mathsf{qFun}(\mathfrak{b},\mathfrak{c})}) &= (F^*)_1(m^{\mathfrak{c}}\hat{G} - G\hat{m}^{\mathfrak{b}}) \\ &= (m^{\mathfrak{c}}\hat{G} - G\hat{m}^{\mathfrak{b}})(\hat{F}) \\ &= m^{\mathfrak{c}}\hat{G}\hat{F} - G\hat{m}^{\mathfrak{b}}\hat{F} \end{aligned}$$

Since F is a cA_∞-functor, we have that $\hat{m}^{\mathfrak{b}}\hat{F} = \hat{F}\hat{m}^{\mathfrak{a}}$, and thus that

$$\begin{aligned} (F^*)_1(\mathfrak{M}_{0,G}^{\mathsf{qFun}(\mathfrak{b},\mathfrak{c})}) &= m^{\mathfrak{c}}\hat{G}\hat{F} - G\hat{F}\hat{m}^{\mathfrak{a}} \\ &= \mathfrak{M}_{0,f^*(G)}^{\mathsf{qFun}(\mathfrak{a},\mathfrak{c})} \end{aligned}$$

For $p = 1$, we find

$$\begin{aligned} (F^*)_1(\mathfrak{M}_1(\eta)) &= (F^*)_1(m^{\mathfrak{c}}\hat{\eta} - \eta\hat{m}^{\mathfrak{b}}) \\ &= (m^{\mathfrak{c}}\hat{\eta} - \eta\hat{m}^{\mathfrak{b}})\hat{F} \\ &= m^{\mathfrak{c}}\hat{\eta}\hat{F} - \eta\hat{F}\hat{m}^{\mathfrak{a}} \\ &= \mathfrak{M}_1((F^*)_1(\eta)) \end{aligned}$$

and for $p \geq 2$, we have by definition of the cA_∞-structure on $\mathsf{qFun}(\mathfrak{a}, \mathfrak{c})$

$$\begin{aligned} (F^*)_1(\mathfrak{M}_p(\eta_p, \dots, \eta_1)) &= (\mathfrak{M}_p(\eta_p, \dots, \eta_1))\hat{F} \\ &= \mathfrak{M}_p(\eta_p\hat{F}, \dots, \eta_1\hat{F}) \\ &= \mathfrak{M}_p((F^*)_1(\eta_p), \dots, (F^*)_1(\eta_1)) \end{aligned}$$

$\square$

Proposition 3.50. *Consider cA_∞-categories $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$. A cA_∞-functor $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ induces a strict cA_∞-functor $F_* : \mathsf{qFun}(\mathfrak{c}, \mathfrak{a}) \longrightarrow \mathsf{qFun}(\mathfrak{c}, \mathfrak{b})$ with underlying morphism $f_*(G) = F * G$, where $*$ is the composition of qA_∞-functors (see Definition 3.7).*

Proof. Let $\eta_i \in \mathsf{qFun}(\mathfrak{c}, \mathfrak{a})(G_{i-1}, G_i)$ be a pre-natural transformation of degree t_i. We define F_* by:

$$(F_*)_k(\eta_k, \dots, \eta_1)(x) = \sum_a (-1)^{\epsilon_a} F(\hat{G}_k(x_a^1), \eta_k(x_a^2), \dots, \eta_1(x_a^{2k}), \hat{G}_0(x_a^{2k+1}))$$

for every $x \in B\mathfrak{c}$ and where

$$\epsilon_a = \sum_{j=1}^{k} \sum_{i=1}^{2j-1} t_j |x_a^i|$$

This is indeed a cA_∞-functor, since we have for $p = 0$

$$\begin{aligned}
(F_*)_1(\mathfrak{M}_{0,G}^{\mathsf{qFun}(\mathfrak{c},\mathfrak{a})}) &= F(\hat{G}, m^{\mathfrak{a}}\hat{G} - G\hat{m}^{\mathfrak{c}}, \hat{G}) \\
&= F(\hat{m}^{\mathfrak{a}}\hat{G} - \hat{G}\hat{m}^{\mathfrak{c}}) \\
&= m^{\mathfrak{b}}\hat{F}\hat{G} - F\hat{G}\hat{m}^{\mathfrak{c}} \\
&= m^{\mathfrak{b}}\widehat{f_*(G)} - f_*(G)\hat{m}^{\mathfrak{c}} \\
&= \mathfrak{M}_{0,f_*(G)}^{\mathsf{qFun}(\mathfrak{c},\mathfrak{b})}
\end{aligned}$$

For $p = 1$, we have by means of the equalities in Proposition 3.25 that

$$\begin{aligned}
(F_*)_1(\mathfrak{M}_1(\eta)) &= (F_*)_1(m^{\mathfrak{a}}\hat{\eta} - \eta\hat{m}^{\mathfrak{c}}) \\
&= F(\hat{G}', m^{\mathfrak{a}}\hat{\eta} - \eta\hat{m}^{\mathfrak{c}}, \hat{G}) \\
&= F(\hat{m}^{\mathfrak{a}}\hat{\eta} - \hat{\eta}\hat{m}^{\mathfrak{c}}) - F(\hat{G}', m^{\mathfrak{a}}\hat{G}' - G'\hat{m}^{\mathfrak{c}}, \hat{G}', \eta, \hat{G}) - F(\hat{G}', \eta, \hat{G}, m^{\mathfrak{a}}\hat{G} - G\hat{m}^{\mathfrak{c}}, \hat{G}) \\
&= m^{\mathfrak{b}}\hat{F}\hat{\eta} - F\hat{\eta}\hat{m}^{\mathfrak{c}} - (F_*)_2(\mathfrak{M}_{0,G'}^{\mathsf{qFun}(\mathfrak{c},\mathfrak{a})}, \eta) - (F_*)_2(\eta, \mathfrak{M}_{0,G}^{\mathsf{qFun}(\mathfrak{c},\mathfrak{a})}) \\
&= \mathfrak{M}_1((F_*)_1(\eta)) - (F_*)_2(\mathfrak{M}_{0,G'}^{\mathsf{qFun}(\mathfrak{c},\mathfrak{a})}, \eta) - (F_*)_2(\eta, \mathfrak{M}_{0,G}^{\mathsf{qFun}(\mathfrak{c},\mathfrak{a})})
\end{aligned}$$

The case for $p \geq 2$ can be proven analogousy to the proof of [3, Theorem-Definition 7.55]. As in the proof of Theorem 3.37, the fact that we are working with qA_∞-functors will yield the appropriate occurrences of the curvature elements $\mathfrak{M}_0$, as it did in the case of $p = 1$. $\square$

We now obtain an extension of [3, Prop 8.49] to the cA_∞-setting:

Proposition 3.51. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be cA_∞-categories, and $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ a cA_∞-homotopy equivalence. Then*

(1) *the functor $F^* : \mathsf{qFun}(\mathfrak{b}, \mathfrak{c}) \longrightarrow \mathsf{qFun}(\mathfrak{a}, \mathfrak{c})$ is a cA_∞-homotopy equivalence, with G^* as its homotopy-inverse.*
(2) *the functor $F_* : \mathsf{qFun}(\mathfrak{c}, \mathfrak{a}) \longrightarrow \mathsf{qFun}(\mathfrak{c}, \mathfrak{b})$ is a cA_∞-homotopy equivalence, with G_* as its homotopy-inverse.*

Proof. Let $F : \mathfrak{a} \longrightarrow \mathfrak{b}$ and $G : \mathfrak{b} \longrightarrow \mathfrak{a}$ be the cA_∞-homotopy equivalence functors. I.e. there are natural transformations $\eta : FG \longrightarrow \mathrm{Id}_{\mathfrak{b}}$ and $\eta' :$

$\mathrm{Id}_{\mathfrak{b}} \longrightarrow FG$ which are homotopic. We thus have pre-natural transformations κ and κ' such that

$$\mathfrak{M}_2(\eta, \eta') - 1_{\mathrm{Id}_{\mathfrak{b}}} = \mathfrak{M}_1(\kappa)$$
$$\mathfrak{M}_2(\eta', \eta) - 1_{FG} = \mathfrak{M}_1(\kappa')$$

where $1_{\mathrm{Id}_{\mathfrak{b}}}$ is the identity transformation from the identity functor to itself.

Let $T : \mathfrak{a} \longrightarrow \mathfrak{c}$ be a qA_∞-functor, we then put

$$\begin{aligned} \mathfrak{e}_0(T) &= (T_*)_1(\eta) \in \mathsf{qFun}(\mathfrak{a}, \mathfrak{c})\Big((f^*g^*)(T), T\Big) \\ \mathfrak{e}_k &= 0 \end{aligned}$$

Due to the fact that η is a natural transformation, so is $\mathfrak{e} : F^*G^* \longrightarrow \mathrm{Id}_{\mathsf{qFun}(\mathfrak{a},\mathfrak{c})}$. Completely analogously one defines the natural transformation $\mathfrak{e}' : \mathrm{Id}_{\mathsf{qFun}(\mathfrak{a},\mathfrak{c})} \longrightarrow F^*G^*$, and the pre-natural transformations $\mathfrak{h} : \mathrm{Id}_{\mathsf{qFun}(\mathfrak{a},\mathfrak{c})} \longrightarrow \mathrm{Id}_{\mathsf{qFun}(\mathfrak{a},\mathfrak{c})}$ and $\mathfrak{h}' : F^*G^* \longrightarrow F^*G^*$, by means of resp. η', κ and κ'.

By definition of the transformations, we now have

$$\overline{\mathfrak{M}}_2(\mathfrak{e}, \mathfrak{e}') - 1_{\mathrm{Id}_{\mathsf{qFun}(\mathfrak{a},\mathfrak{c})}} = \overline{\mathfrak{M}}_1(\mathfrak{h})$$
$$\overline{\mathfrak{M}}_2(\mathfrak{e}', \mathfrak{e}) - 1_{F^*G^*} = \overline{\mathfrak{M}}_1(\mathfrak{h}')$$

where $\overline{\mathfrak{M}}$ is the structure on $\mathsf{cFunc}\big(\mathsf{qFun}(\mathfrak{a}, \mathfrak{c}), \mathsf{qFun}(\mathfrak{a}, \mathfrak{c})\big)$.

We have thus proven that F^*G^* is homotopic to the identity. Analogously one shows that G^*F^* is homotopic to the identity, and thus that $\mathsf{qFun}(\mathfrak{a}, \mathfrak{c})$ is cA_∞-homotopy equivalent to $\mathsf{qFun}(\mathfrak{b}, \mathfrak{c})$. The other cA_∞-homotopy equivalence is proven analogously by means of the functors F_* and G_*. □

Remark 3.52. Since, in the proof of Proposition 3.51, the functors F^* and G^* are in fact strict cdg-functors and the (pre-)natural transformations $\mathfrak{e}$, $\mathfrak{e}'$, $\mathfrak{h}$, $\mathfrak{h}'$ only have a 0-component, the cA_∞-homotopy equivalence between $\mathsf{qFun}(\mathfrak{a}, \mathfrak{c})$ and $\mathsf{qFun}(\mathfrak{b}, \mathfrak{c})$ is in fact a cA_∞-homotopy equivalence by means of strict cdg-functors.

Proposition 3.53. *Take $F, G : \mathfrak{a} \longrightarrow \mathfrak{b}$ homotopic cA_∞-functors between cA_∞-categories. For every cA_∞-functor $K : \mathfrak{a} \longrightarrow \mathfrak{b}$, we have that*

$$H^*(\mathit{cFunc}(\mathfrak{a}, \mathfrak{b})(F, K)) \cong H^*(\mathit{cFunc}(\mathfrak{a}, \mathfrak{b})(G, K))$$

Proof. Consider the natural transformation $\eta : G \longrightarrow F$ and $\mu : F \longrightarrow G$ expressing the homotopy. We define the maps (well-defined due to the fact that η and μ are natural transformations)

$$u : H^*(\mathsf{cFunc}(\mathfrak{a}, \mathfrak{b})(F, K)) \longrightarrow H^*(\mathsf{cFunc}(\mathfrak{a}, \mathfrak{b})(G, K)) : \rho \mapsto \mathfrak{M}_2(\rho, \eta)$$

$$v : H^*(\mathsf{cFunc}(\mathfrak{a}, \mathfrak{b})(G, K)) \longrightarrow H^*(\mathsf{cFunc}(\mathfrak{a}, \mathfrak{b})(F, K)) : \rho \mapsto \mathfrak{M}_2(\rho, \mu)$$

Since $\mathfrak{M}_2(\mathfrak{M}_2 \otimes 1) = \mathfrak{M}_2(1 \otimes \mathfrak{M}_2) - \mathfrak{M}_3(1^{\otimes 2} \otimes \mathfrak{M}_1) - \mathfrak{M}_3(1 \otimes \mathfrak{M}_1 \otimes 1) - \mathfrak{M}_3(\mathfrak{M}_1 \otimes 1^{\otimes 2}) - \mathfrak{M}_1(\mathfrak{M}_3)$, it is clear that

$$\begin{aligned}(vu)(\rho) &= \mathfrak{M}_2(\rho, \mathfrak{M}_2(\eta, \mu)) - \mathfrak{M}_1(\mathfrak{M}_3(\rho, \eta, \mu))\\ &= \mathfrak{M}_2(\rho, \mathrm{Id}_F + \mathfrak{M}_1(k)) - \mathfrak{M}_1(\mathfrak{M}_3(\rho, \eta, \mu))\\ &= \rho + \mathfrak{M}_1(\mathfrak{M}_2(\rho, k)) - \mathfrak{M}_2(\mathfrak{M}_1(\rho), k) - \mathfrak{M}_1(\mathfrak{M}_3(\rho, \eta, \mu))\\ &= \rho + \mathfrak{M}_1\big(\mathfrak{M}_2(\rho, k) - \mathfrak{M}_3(\rho, \eta, \mu)\big)\end{aligned}$$

Analogously, one proves that uv =Id, completing the proof of the announced equivalence. □

3.6. **Trivialization of cA_∞-equivalences.** In general, in the unfiltered setting, the notions of homotopic cA_∞-functors and cA_∞-homotopy equivalences are known to trivialize the theory of cA_∞-categories in the sense that too many categories become cA_∞-homotopy equivalent. We first discuss the unfiltered case, before turning our attention to the uniformly weakly curved case.

3.6.1. *Unfiltered setting.* The following results, mostly due to Kontsevich, can be found in [18, Remark 7.3].

Let us consider cA_∞-algebras $(\mathfrak{a}, m)$ and $(\mathfrak{b}, m')$ over a field k with nonzero curvature elements that do not belong to the one-dimensional vector subspaces generated by the units of $\mathfrak{a}$ and $\mathfrak{b}$ respectively. In this case, we can extend an isomorphism of graded vector spaces $f : \mathfrak{a} \longrightarrow \mathfrak{b}$, preserving the units and curvature, to a cA_∞-homotopy equivalence $F = (f_0, f_1, f_2, \ldots) : \mathfrak{a} \longrightarrow \mathfrak{b}$ with $f_0 = 0$ and $f_1 = f$.

This result implies the following:

Theorem 3.54. [10, Theorem 2.1] *If $(\mathfrak{a}, m)$ is a cA_∞-algebra for which the curvature $m_0 = c$ is nonzero, then $(\mathfrak{a}, m)$ is cA_∞-homotopy equivalent to $(\mathfrak{a}, m')$, where $m'_0 = c$ and all higher multiplications $m'_i = 0$ for $i > 0$.*

If we consider the extension of $Id_{\mathfrak{a}}$ used in the previous theorem, we see that the associated functor F is defined by the identities

$$\begin{aligned} m'_0 &= m_0\\ 0 = m'_1(-) &= m_1(-) + f_2(-, m_0) - f_2(m_0, -) \qquad (24)\\ 0 = m'_1(f_2) + m'_2(-,-) &= m_2(-,-) - f_2(-, m_1) - f_2(m_1, -)\\ &\quad + f_3(m_0, -, -) + f_3(-, m_0, -) + f_3(-, -, m_0)\\ &\vdots \end{aligned}$$

For the convenience of the reader and for further reference, we briefly sketch a possible approach to the trivialization result. Consider the restriction d_0 of the differential $\hat{m}$ on $B\mathfrak{a}$, to the components determined by m_0. We see that this is again a differential, and that $(B\mathfrak{a}, d_0)$ is acyclic. Let us now denote $d_n = \hat{m}_0 + \sum_{k \geq n+1} \hat{m}_k$. Writing out the conditions, one finds that there are coalgebra isomorphisms

$$F_n : (B\mathfrak{a}, d_n) \longrightarrow (B\mathfrak{a}, d_{n+1})$$

with $f_1 = \mathrm{Id}_{\mathfrak{a}}$ and $f_k = 0$ for $k \neq n+1$, commuting with the derivations d_k. As such we obtain a cA_∞-homotopy equivalence $(\mathfrak{a}, m) \cong (\mathfrak{a}, m')$.

When we inspect these functors F_n in more detail, we see that e.g. $F_0 : (B\mathfrak{a}, d) \longrightarrow (B\mathfrak{a}, d_1)$ is defined by the identities

$$\begin{aligned} m_0 &= m_0 \\ 0 &= m_1(-) + f_2(-, m_0) - f_2(m_0, -) \\ m_2(-,-) &= m_2(-,-) - f_2(-, m_1) - f_2(m_1, -) \\ &\vdots \end{aligned} \tag{25}$$

The previous defining identities show how the components m_n can be constructed by means of m_0 and the higher components of a functor which has the identity as its first component, leading to the trivialization.

3.6.2. *Uniformly weakly curved setting.* In this section, we consider uniformly weakly curved filtered cA_∞-categories and we explain to what extent the trivialization discussed in §3.6 can be avoided.

Let $\mathfrak{a}$ be uniformly weakly curved with $m_0 \in \mathcal{F}^l\mathfrak{a}$ for a certain $l > 0$. Looking at the identities (24) and (25), we see that they are now subject to filtration restraints. Since the multiplications are filtered, we have $m_k(\mathcal{F}^\lambda \mathfrak{a}^{\otimes k}) \subset \mathcal{F}^\lambda \mathfrak{a}$, and since the functor F is filtered we have $F_k(\mathcal{F}^\lambda \mathfrak{a}^{\otimes k}) \subset \mathcal{F}^\lambda \mathfrak{a}$. Applying this for instance to the identity

$$m_1(-) = f_2(m_0, -) - f_2(-, m_0)$$

we see that if it is fulfilled, we have $m_1(\mathcal{F}^\lambda \mathfrak{a}) \subset \mathcal{F}^{\lambda+l}\mathfrak{a}$. Analogously we obtain similar results for the higher multiplications. As such it is clear that the required functors from the previous section generally do not exist in the setting of uniformly weakly curved cA_∞-categories.

Of course, this does not yet prove that trivialization cannot occur in the setting of uniformly weakly curved cA_∞-categories. However, the following observation places a strong restraint upon the possibility of trivialization.

Let $(\mathfrak{a}, m)$ be an l-curved cA_∞-category, and let $(\mathfrak{a}, m')$ be the cA_∞-category with the same underlying quiver $\mathfrak{a}$, $m'_0 = m_0$ and $m'_i = 0$ for all $i > 0$. Suppose there is a cA_∞-homotopy equivalence

$$F : (\mathfrak{a}, m) \longrightarrow (\mathfrak{a}, m').$$

Since F, its homotopy-inverse and the pre-natural transformations expressing the homotopy are all filtered, we obtain an induced A_∞-homotopy equivalence

$$\tilde{F} : (\mathfrak{a}/\mathcal{F}^l\mathfrak{a}, \tilde{m}) \longrightarrow (\mathfrak{a}/\mathcal{F}^l\mathfrak{a}, \tilde{m}') = (\mathfrak{a}/\mathcal{F}^l\mathfrak{a}, 0).$$

Hence, l-curved cA_∞-categories with a non-trivial A_∞-quotient are effectively protected against trivialization. This explains why the Fukaya type cA_∞-categories (see Example 3.14) and the wcA_∞-categories of Positselski (see Example 3.13) give rise to non-trivial theories.

4. The curved Yoneda-Lemma

One of the standout features of the curved world is that in general, it is not possible to construct representable cdg-modules over a cdg-category. This situation was remedied by Polishchuk and Positselski in [19] by relaxing the notion of cdg-modules to that of qdg-modules. Using this notion, it is not hard to obtain a Yoneda type embedding, as explained in §4.1.

The main goal of this section, which is adressed in §4.2, is to present a cA_∞-version of the Yoneda Lemma, inspired upon Fukaya's treatment of the A_∞-case in [3, §9]. To do so, we define the relevant module category $\mathsf{Mod}_{q\infty}(\mathfrak{a})$ as the functor category of strict qA_∞-functors from $\mathfrak{a}^{\mathrm{op}}$ to the cdg category of precomplexes of k-modules. In particular, $\mathsf{Mod}_{q\infty}(\mathfrak{a})$ is itself a cdg category, and inside we can define the full cA_∞-subcategory of representable modules $\mathsf{Rep}_{q\infty}(\mathfrak{a})$. In our main Theorem 4.15, we obtain a cA_∞-homotopy equivalence $Y : \mathfrak{a} \longrightarrow \mathsf{Rep}_{q\infty}(\mathfrak{a})$.

It is well known that in general, it is impossible to endow the tensor quiver $\mathfrak{a} \otimes \mathfrak{b}$ of two A_∞-categories with a natural A_∞-structure. On the other hand, as soon as one of the tensor factors is itself a dg category, a natural tensor structure does exist. In §4.3, as an application of Theorem 4.15, we define the *Yoneda tensor product* of two cA_∞-categories by means of the standard tensor product of the cdg categories of representable modules. The main result of this section is the construction of a cA_∞-structure on the tensor product quiver of a cdg category and a cA_∞-category in Proposition 4.19, which is proven in Theorem 4.26 to be homotopy equivalent to the Yoneda tensor product.

4.1. **qdg-modules and Yoneda.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be cdg-categories. Recall from Definition 3.17 that a qdg-functor from $\mathfrak{a}$ to $\mathfrak{b}$ with underlying map $f : \mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathfrak{b})$ consists of the same datum $F \in \mathbf{C}^1(\mathfrak{a}, \mathfrak{b})_f$ as a cdg-functor, but from the conditions (21), (22), and (23), condition (21) is omitted.

Definition 4.1. [19, §1.4] A *qdg-module* over a cdg-category $\mathfrak{a}$ is given by a strict qdg-functor from $\mathfrak{a}^{\mathrm{op}}$ to the cdg-category $\mathsf{PCom}(k)$ of precomplexes of k-modules (see Example 2.54). Similarly, a *cdg-module* over $\mathfrak{a}$ is a strict cdg-functor from $\mathfrak{a}^{\mathrm{op}}$ to $\mathsf{PCom}(k)$.

We thus know that a cdg-module (resp. qdg-module) M is given by an underlying map

$$\mathrm{Ob}(\mathfrak{a}) \longrightarrow \mathrm{Ob}(\mathsf{PCom}(k)) : A \longmapsto M(A)$$

and k-linear maps

$$M_{A,A'} : \mathfrak{a}(A, A') \longrightarrow \mathrm{Hom}(M(A'), M(A)) : f \longmapsto M(f)$$

fulfilling the conditions (21), (22), and (23) (resp. (22) and (23)).

For qdg-modules M and N, we put $\mathrm{Hom}(M,N) \subseteq \prod_{A \in \mathfrak{a}} \mathrm{Hom}(M(A), N(A))$ the graded k-module of natural transformations, where a natural transformation of degree n is given by a collection $(\rho_A)_{A\in\mathfrak{a}}$ with $\rho_A \in \mathrm{Hom}^n(M(A), N(A))$ with for all $f \in \mathfrak{a}(A, A')$:

$$m_2'(\rho_{A'}, M(f)) = (-1)^{n|f|} m_2'(N(f), \rho_A)$$

This defines the quiver $\mathsf{Mod}_{qdg}(\mathfrak{a})$ of qdg-modules over $\mathfrak{a}$, and we know by Remark 3.38 that it inherits a cdg-structure from the qdg-functor category.

We denote the cdg-structure on $\mathfrak{a}$ by m and the one on $\mathsf{PCom}(k)$ by m^{PCom}. Since the pre-natural transformations between qdg-modules only have a 0-component, and the qdg-modules are strict qdg-functors, we know that the cdg-structure $\mathfrak{M}$ on $\mathsf{Mod}_{qdg}(\mathfrak{a})$ is such that:

- $\mathfrak{M}_2$ is the composition of natural transformations based upon m_2^{PCom};

- $((\mathfrak{M}_1)_{M,N})_A = (m_1^{\mathsf{PCom}})_{M(A),N(A)}$;
- $((\mathfrak{M}_0)_M)_A = (m_0^{\mathsf{PCom}})_{M(A)} - M((m_0)_A)$.

Clearly, if we let $\mathsf{Mod}_{cdg}(\mathfrak{a})$ denote the dg-category of cdg-modules on $\mathfrak{a}$, we have

$$(\mathsf{Mod}_{qdg}(\mathfrak{a}))_\infty = \mathsf{Mod}_{cdg}(\mathfrak{a})$$

Lemma 4.2. *Every object A in a cA_∞-category $\mathfrak{a}$ determines a representable qdg-module with underlying morphism*

$$\mathfrak{a}(-,A) : \mathfrak{a}^{\mathrm{op}} \longrightarrow \mathsf{PCom}(k) : B \longmapsto (\mathfrak{a}(B,A), (m_1)_{B,A})$$

and given by

$$\mathfrak{a}(-,A) : \mathfrak{a}(B,B') \longrightarrow \mathrm{Hom}(\mathfrak{a}(B',A), \mathfrak{a}(B,A)) : f \longmapsto \mathfrak{a}(f,A) = m_2(-,f)$$

Proof. Consider $f \in \mathfrak{a}(B,B'), g \in \mathfrak{a}(B',B'')$. We then have

(1) $\mathfrak{a}(-,A)(m_0^B) = m_2(-,m_0^B) \neq m_1(m_1) = m^{\mathsf{PCom}}_{0,\mathfrak{a}(B,A)}$
(2) $\mathfrak{a}(-,A)(m_1(f)) = m_2(-,m_1(f)) = m_1(m_2(-,f)) - m_2(m_1,f) = m_1^{\mathsf{PCom}}(\mathfrak{a}(f,A))$
(3) $\mathfrak{a}(-,A)(m_2(g,f)) = m_2(-,m_2(g,f)) = m_2(m_2(-,g),f) = m_2^{\mathsf{PCom}}(\mathfrak{a}(-,A)(f), \mathfrak{a}(-,A)(g))$

□

Note that in general, the representable qdg-modules fail to be cdg-modules. Using the representable qdg-modules, we obtain a Yoneda embedding:

Proposition 4.3. *There is a fully faithful strict cdg-embedding*

$$Y = Y^{\mathfrak{a}}_{qdg} : \mathfrak{a} \longrightarrow \mathsf{Mod}_{qdg}(\mathfrak{a}) : A \longmapsto \mathfrak{a}(-,A),$$

$$Y : \mathfrak{a}(A,A') \longrightarrow \mathrm{Hom}(\mathfrak{a}(-,A), \mathfrak{a}(-,A')) : g \longmapsto (m_2(g,-))_{B\in\mathfrak{a}}$$

Proof. The existence of the fully faithful embedding is based upon the Yoneda Lemma for the underlying $\mathbb{Z}$-graded k-linear categories. One verifies that the resulting functor satisfies the cdg-axioms. By definition of the multiplications on $\mathsf{Mod}_{qdg}(\mathfrak{a})$ we have

(1) $Y(m_0) = m_2(m_0,-) = m_1(m_1) + m_2(-,m_0) = \mathfrak{M}_0$.
(2) $Y(m_1) = m_2(m_1,-) = m_1(m_2(-,-)) - m_2(-,m_1) = \mathfrak{M}_1(Y)$.
(3) $Y(m_2) = m_2(m_2(-,-),-) = m_2(-,m_2(-,-)) = \mathfrak{M}_2(Y,Y)$

where the second equality in (3) comes from the fact that there are no higher order multiplications. □

Definition 4.4. We define the *category of representable qdg-modules*,

$$\mathsf{Rep}_{qdg}(\mathfrak{a}) \subseteq \mathsf{Mod}_{qdg}(\mathfrak{a})$$

as the full cdg-subcategory consisting of the objects $\{\mathfrak{a}(-,A) | A \in \mathfrak{a}\}$.

We thus obtain a strong equivalence of cdg categories $Y : \mathfrak{a} \longrightarrow \mathsf{Rep}_{qcd}(\mathfrak{a})$.

4.2. qA_∞**-modules and Yoneda.** In this section, we consider a stricly unital cA_∞-category $(\mathfrak{a}, m)$, and the category $\mathsf{PCom}(k)$ of filtered precomplexes (see Example 2.54). We denote the multiplicative structure of $\mathfrak{a}$ by m, whereas we denote the multiplications of $\mathsf{PCom}(k)$ by m^{PCom}.

The category $\mathsf{PCom}(k)$ can be endowed with a natural filtration induced by the filtered precomplexes. More precisely, for precomplexes M, N, we put

$$\mathcal{F}^l\mathsf{PCom}(M,N) = \{f \in \mathsf{PCom}(M,N) | \forall p \in \mathbb{L} : f(\mathcal{F}^p M) \subset \mathcal{F}^{p+l}N\}.$$

Definition 4.5. A *qA_∞-module* over $\mathfrak{a}$ is a strict qA_∞-functor $\mathfrak{a}^{\mathrm{op}} \longrightarrow \mathsf{PCom}(k)$. A *$cA_\infty$-module* over $\mathfrak{a}$ is a strict cA_∞-functor $\mathfrak{a}^{\mathrm{op}} \longrightarrow \mathsf{PCom}(k)$.

We denote by

$$\mathsf{Mod}_{q\infty}(\mathfrak{a}) \subseteq \mathsf{qFun}(\mathfrak{a}^{\mathrm{op}}, \mathsf{PCom}(k))$$

the full cA_∞-subcategory of qA_∞-modules and by

$$\mathsf{Mod}_{c\infty}(\mathfrak{a}) \subseteq \mathsf{Mod}_{q\infty}(\mathfrak{a})$$

the full cA_∞-subcategory of cA_∞-modules. Hence, the curvature of a qA_∞-module F is given by $\mathfrak{M}_0^F = p_1(\hat{m}^{\mathsf{PCom}}\hat{F} - \hat{F}\hat{m})$.

Remark 4.6. By Remark 3.38, $\mathsf{Mod}_{q\infty}(\mathfrak{a})$ is in fact a cdg-category, and $\mathsf{Mod}_{c\infty}(\mathfrak{a})$ is a dg-category.

Using qA_∞-modules, we can construct representable modules over $\mathfrak{a}$.

Definition 4.7. Let $\mathfrak{a}$ be a cA_∞-category, and consider an object A in $\mathfrak{a}$. The *representable module* $y(A)$ is given by the qA_∞-functor

$$y(A) : \mathfrak{a}^{\mathrm{op}} \longrightarrow \mathsf{PCom}(k)$$

with underlying map

$$\mathrm{Ob}(\mathfrak{a}^{\mathrm{op}}) \longrightarrow \mathrm{Ob}(\mathsf{PCom}(k)) : B \mapsto (\mathfrak{a}(B,A), m_1^{\mathfrak{a}})$$

and components

$$(y(A)_1)_{B,C} : \mathfrak{a}(B,C) \longrightarrow \mathsf{PCom}(k)\big(\mathfrak{a}(C,A), \mathfrak{a}(B,A)\big) : f \mapsto m_2(-,f);$$
$$(y(A)_2)_{B,C} : \mathfrak{a}^{\otimes 2}(B,C) \longrightarrow \mathsf{PCom}(k)\big(\mathfrak{a}(C,A), \mathfrak{a}(B,A)\big) : (g,f) \mapsto m_3(-,g,f);$$
$$\vdots$$

Remark 4.8. The components of a representable module $y(A)$ are filtered with respect to the filtration on $\mathsf{PCom}(k)$. Indeed, for $(f_n, \dots, f_1) \in \mathcal{F}^l\mathfrak{a}^{\otimes n}(B,C)$, we have $m_{n+1}(-, f_n, \dots, f_1) \in \mathcal{F}^l\mathsf{PCom}(k)\big(y(A)(C), y(A)(B)\big)$ since m is filtered.

Next, we calculate the curvature $\mathfrak{M}_0^{y(A)} = p_1\big(\hat{m}\widehat{y(A)} - \widehat{y(A)}\hat{m}\big)$ of the representable modules $y(A)$. By Proposition 3.27, the representable module $y(A)$ is a cA_∞-module if and only if $\mathfrak{M}_0^{y(A)} = 0$.

Lemma 4.9. *The curvature* $\mathfrak{M}_0^{y(A)}$ *of the representable module* $y(A)$ *is given by the components*

$$(\mathfrak{M}_0^{y(A)})_0 = m_2(m_0,-) \in \mathit{PCom}(k)\big(\mathfrak{a}(B,A),\mathfrak{a}(B,A)\big);$$
$$(\mathfrak{M}_0^{y(A)})_1 : \mathfrak{a}(B,C) \longrightarrow \mathit{PCom}(k)\big(\mathfrak{a}(C,A),\mathfrak{a}(B,A)\big) : f \mapsto m_3(m_0,-,f);$$
$$(\mathfrak{M}_0^{y(A)})_2 : \mathfrak{a}^{\otimes 2}(B,C) \longrightarrow \mathit{PCom}(k)\big(\mathfrak{a}(C,A),\mathfrak{a}(B,A)\big) : (g,f) \mapsto m_4(m_0,-,g,f);$$
$$\vdots$$

In particular, if $m_0 \in \mathcal{F}^l\mathfrak{a}$*, it follows that* $\mathfrak{M}_0^{y(A)} \in \mathcal{F}^l\mathsf{Mod}_{q\infty}(\mathfrak{a})$*.*

Proof. Writing out the expression $p_1\big(\hat{m}\widehat{y(A)} - \widehat{y(A)}\hat{m}\big)$, we find for $p=0$

$$\begin{aligned}\big(m^{\mathsf{PCom}}(\widehat{y(A)}) - y(A)(\hat{m})\big)_0 &= m_0^{\mathsf{PCom}} + (y(A))_1(m_0)\\ &= m_0^{\mathsf{PCom}} + m_2(-,m_0)\\ &= m_1m_1 + m_2(-,m_0)\\ &= m_2(m_0,-)\end{aligned}$$

for $p=1$

$$\begin{aligned}\Big(m^{\mathsf{PCom}}(\widehat{y(A)}) - y(A)(\hat{m})\Big)_1(f) &= m_1^{\mathsf{PCom}}(y(A)_1(f)) - y(A)_1(m_1(f)) - y(A)_2(f,m_0) + y(A)_2(m_0,f)\\ &= m_1(m_2(-,f)) - m_2(m_1(-),f) - m_2(-,m_1(f)) - m_3(-,f,m_0) + m_3(-,m_0,f)\\ &= m_3(m_0,-,f)\end{aligned}$$

where the equalities are due to the cA_∞-identities of the multiplication m on $\mathfrak{a}$.

Analogously, one proves the cases for $p \geq 2$. □

Remark 4.10. Lemma 4.9 shows that if A has non-zero curvature m_0, in general none of the cA_∞-functor identities needs to be fulfilled by $y(A)$. Unlike in the cdg case, where a qdg-functor still satisfies some structure compatibility, we thus observe that the full relaxation of compatibility requirements in the definition of qA_∞-functors is necessary in order to capture representable modules.

We now introduce a special kind of pre-natural transformations between representable modules $y(A)$, $y(B)$, with A, B objects in $\mathfrak{a}$.

Definition 4.11. For $(f_n,\dots,f_1) \in \mathfrak{a}^{\otimes n}(A,B)$, we define the pre-natural transformation

$$m(f_n,\dots,f_1,-) : y(A) \longrightarrow y(B)$$

by the components

$$\big(m(f_n,\dots,f_1,-)\big)_0 = m_{n+1}(f_n,\dots,f_1,-) \in \mathsf{PCom}(k)\big(\mathfrak{a}(C,A),\mathfrak{a}(C,B)\big);$$
$$\begin{aligned}\big(m(f_n,\dots,f_1,-)\big)_1 : \mathfrak{a}(C,D) &\longrightarrow \mathsf{PCom}(k)\big(\mathfrak{a}(D,A),\mathfrak{a}(C,B)\big)\\ g &\longmapsto m_{n+2}(f_n,\dots,f_1,-,g);\end{aligned}$$
$$\begin{aligned}\big(m(f_n,\dots,f_1,-)\big)_2 : \mathfrak{a}^{\otimes 2}(C,D) &\longrightarrow \mathsf{PCom}(k)\big(\mathfrak{a}(D,A),\mathfrak{a}(C,B)\big)\\ (g_2,g_1) &\longmapsto m_{n+3}(f_n,\dots,f_1,-,g_2,g_1);\end{aligned}$$
$$\vdots$$

Remark 4.12. As with the definition of representable modules, the components of this pre-natural transformation are readily seen to be filtered (see Remark 4.8).

Having introduced these pre-natural transformations between representable qA_∞-modules, we are now able to construct a cA_∞-version of the Yoneda embedding.

Proposition 4.13. *Let* $(\mathfrak{a}, m)$ *be a* cA_∞*-category. There exists a strict* cA_∞*-functor*

$$Y = Y^{\mathfrak{a}}_{q\infty} : \mathfrak{a} \longrightarrow \mathsf{Mod}_{q\infty}(\mathfrak{a})$$

given by the underlying map $A \mapsto y(A)$ *and*

$$(Y_1)_{A,B} : \mathfrak{a}(A,B) \longrightarrow \mathsf{Mod}_{q\infty}(\mathfrak{a})(y(A), y(B)) : f \mapsto m(f,-);$$
$$(Y_2)_{A,B} : \mathfrak{a}^{\otimes 2}(A,B) \longrightarrow \mathsf{Mod}_{q\infty}(\mathfrak{a})(y(A), y(B)) : (g,f) \mapsto m(g,f,-);$$
$$\vdots$$

Proof. Analogously to Remarks 4.8 and 4.12, the components of the Yoneda functor are seen to be filtered.

We will check the cA_∞-functor identities for Y. By Lemma 4.9, we have for $p = 0$ that

$$Y_1(m_0) = m(m_0, -) = (\mathfrak{M}_0)_{Y(A)}.$$

For $p = 1$, we have to check for every $f \in \mathfrak{a}(A,B)$ that the natural transformations

$$\Big(Y_1(m_1) + Y_2(-, m_0) - Y_2(m_0, -)\Big)(f) = m(m_1(f), -) + m(f, m_0, -) - m(m_0, f, -)$$

and

$$\begin{aligned}\Big(\mathfrak{M}_1(Y_1)\Big)(f) &= \mathfrak{M}_1(m(f,-))\\ &= m_1^{\mathsf{PCom}}(m(f,-)) + m_2^{\mathsf{PCom}}(y(B), m(f,-)) + m_2^{\mathsf{PCom}}(m_2(f,-), y(A)) - m(f,-,\hat{m})\end{aligned}$$

are equal. We also check this component-wise.

For $q = 0$, we have that

$$\Big(m(m_1(f),-)+m(f,m_0,-)-m(m_0,f,-)\Big)_0 = m_2(m_1(f),-)+m_3(f,m_0,-)-m_3(m_0,f,-)$$

$$\begin{aligned}\Big((\mathfrak{M}_1(Y_1))(f)\Big)_0 &= m_1^{\mathsf{PCom}}(m_2(f,-)) - m_3(f,-,m_0)\\ &= m_1(m_2(f,-)) - m_2(f, m_1(-)) - m_3(f,-,m_0)\end{aligned}$$

By the cA_∞-identities for the multiplication m on $\mathfrak{a}$, we know that they are equal.

For $q = 1$, we have for every $g \in \mathfrak{a}(C,D)$ that

$$\Big(m(m_1(f),-)+m(f,m_0,-)-m(m_0,f,-)\Big)_1(g) = m_3(m_1(f),-,g)+m_4(f,m_0,-,g)-m_4(m_0,f,-,g)$$

$$\begin{aligned}\Big((\mathfrak{M}_1(Y_1))(f)\Big)_1(g) &= m_1^{\mathsf{PCom}}(m_3(f,-,g)) + m_2^{\mathsf{PCom}}(m_2(-,g), m_2(f,-))) + m_2^{\mathsf{PCom}}(m_2(f,-), m_2(-,g))\\ &\quad - m_3(f,-,m_1(g)) + m_4(f,-,m_0,g) - m_4(f,-,g,m_0)\\ &= m_1(m_3(f,-,g)) - m_3(f, m_1(-), g) + m_2(m_2(f,-), g) + m_2(f, m_2(-,g))\\ &\quad - m_3(f,-,m_1(g)) + m_4(f,-,m_0,g) - m_4(f,-,g,m_0)\end{aligned}$$

The equality is again due to the cA_∞-identities for the multiplication m on $\mathfrak{a}$

Analogously, one finds the equality for $q \geq 2$.

In a similar fashion, one shows, by means of the cA_∞-structure on $\mathsf{Mod}_{q\infty}(\mathfrak{a})$ (which is actually a cdg-structure) and the cA_∞-structure of $\mathfrak{a}$, that the cA_∞-functor identities also hold for $p \geq 2$. $\square$

Definition 4.14. The *category of representable qA_∞-modules* is the full cA_∞-subcategory $\mathsf{Rep}_{q\infty}(\mathfrak{a}) \subseteq \mathsf{Mod}_{q\infty}(\mathfrak{a})$ consisting of the objects $\{y(A) | A \in \mathfrak{a}\}$.

We are now ready to prove our main theorem, which extends the situation for A_∞-categories [3, §9]. The constructions in the proof are inspired upon [3, §8].

Theorem 4.15. *Let $(\mathfrak{a}, m)$ be a strictly unital cA_∞-category. The Yoneda functor $Y : \mathfrak{a} \longrightarrow \mathsf{Rep}_{q\infty}(\mathfrak{a})$ is a cA_∞-homotopy equivalence. For $l \in \mathbb{L}$, Y restricts to a cA_∞-homotopy equivalence $Y_l : \mathfrak{a}_l \longrightarrow \mathsf{Rep}_{q\infty}(\mathfrak{a})_l$ between the full cA_∞-subcategories of l-curved objects.*

In particular, every (l-curved) cA_∞-category is canonically cA_∞-homotopy equivalent to an (l-curved) cdg category.

Proof. The second statement easily follows from the first one by Lemma 4.9.

Put $\mathsf{Rep}(\mathfrak{a}) = \mathsf{Rep}_{q\infty}(\mathfrak{a})$. We will construct a homotopy inverse $\Pi : \mathsf{Rep}(\mathfrak{a}) \longrightarrow \mathfrak{a}$ of Y . We define the strict cA_∞-functor Π with underlying map $y(A) = (\mathfrak{a}(-, A), m_1^{\mathfrak{a}}) \mapsto A$ and components

$(\Pi_1)_{A,B} : \mathsf{Rep}(\mathfrak{a})(y(A), y(B)) \longrightarrow \mathfrak{a}(A, B) : \eta \mapsto \eta_0(1_A);$

$(\Pi_2)_{A,B} : \mathsf{Rep}(\mathfrak{a})^{\otimes 2}(y(A), y(B)) \longrightarrow \mathfrak{a}(A, B) : (\eta, \mu) \mapsto \big(\eta_1(\mu_0(1))\big)(1);$

$(\Pi_3)_{A,B} : \mathsf{Rep}(\mathfrak{a})^{\otimes 3}(y(A), y(B)) \longrightarrow \mathfrak{a}(A, B) : (\eta, \mu, \xi) \mapsto \big(\eta_2(\mu_0(1), \xi_0(1))\big)(1);$

$\vdots$

$(\Pi_k)_{A,B} : \mathsf{Rep}(\mathfrak{a})^{\otimes k}(y(A), y(B)) \longrightarrow \mathfrak{a}(A, B) :$

$$(\eta^{(k)}, \dots, \eta^{(1)}) \longmapsto \Big((\eta^{(k)})_{k-1}\Big((\eta^{(k-1)})_0(1), \dots, (\eta^{(1)})_0(1)\Big)\Big)(1)$$

for $k \geq 1$. Here, the unit elements, denoted by 1, should be understood in the appropriate way. For instance, if we consider

$$(\eta, \mu, \xi) \in \mathsf{Rep}(\mathfrak{a})^{\otimes 3}(y(A), y(B)) = \prod_{C,D \in \mathfrak{a}} \mathsf{Rep}(\mathfrak{a})\Big(y(D), y(B)\Big) \otimes \mathsf{Rep}(\mathfrak{a})\Big(y(C), y(D)\Big) \otimes \mathsf{Rep}(\mathfrak{a})\Big(y(A), y(C)\Big)$$

then $\Pi_3(\eta, \mu, \xi) = \Big(\eta_2\big(\mu_0(1_C), \xi_0(1_A)\big)\Big)(1_D)$. In the rest of the proof, we will stick to the shorthand notation 1. We first show that the components of the functor Π are indeed filtered. For $k \geq 1$, consider

$$(\eta^{(k)}, \dots, \eta^{(1)}) \in \mathcal{F}^l \mathsf{Rep}(\mathfrak{a})^{\otimes k}(y(A), y(B)) = \Big(\bigcup_{l_1 + \dots + l_k = l} \mathcal{F}^{l_k} \mathsf{Rep}(\mathfrak{a}) \otimes \dots \otimes \mathcal{F}^{l_1} \mathsf{Rep}(\mathfrak{a})\Big)(y(A), y(B)).$$

By definition of the filtration on $\mathsf{Rep}(\mathfrak{a})$ we have

$$\Big((\eta^{(k-1)})_0(1), \dots, (\eta^{(1)})_0(1)\Big) \in \mathcal{F}^{l_{k-1} + \dots + l_1} \mathfrak{a}^{\otimes k-1}$$

and

$$\Big((\eta^{(k)})_{k-1}\Big((\eta^{(k-1)})_0(1),\dots,(\eta^{(1)})_0(1)\Big)\Big)(1)\in\mathcal{F}^{l_k+\dots+l_1}\mathfrak{a}$$

proving that Π_k is filtered.

We will now show that Π is indeed a cA_∞-functor. We have for $p=0$

$$m_0=\big(m(m_0,-)\big)_0(1)=\Pi_1(\mathfrak{M}_0^{y(A)})$$

and, using the strict unitality of $\mathfrak{a}$, for $p=1$

$$\begin{aligned}
&\big[\Pi_1(\mathfrak{M}_1)+\Pi_2(-,\mathfrak{M}_0^{y(A)})-\Pi_2(\mathfrak{M}_0^{y(A)},-)\big](\eta)\\
&\quad=\Big[\Pi_1(m^{\mathsf{PCom}}\hat{\eta}-\eta\hat{m})\Big]+\Pi_2(\eta,\mathfrak{M}_0^{y(A)})-\Pi_2(\mathfrak{M}_0^{y(A)},\eta)\\
&\quad=(m^{\mathsf{Pcom}}(\eta_0))(1)-(\eta_1(m_0))(1)+(\eta_1(m_0))(1)-m_3(m_0,1,\eta_0(1))\\
&\quad=m_1(\eta_0(1))-\eta_0(m_1(1))\\
&\quad=m_1(\eta_0(1))\\
&\quad=[m_1(\Pi_1)]\,(\eta)
\end{aligned}$$

Analogously, one checks the identities for $p\geq 2$.

By Lemma 4.16, we have $\Pi*Y=\mathrm{Id}_{\mathfrak{a}}$. In order to show that Y and Π are cA_∞-homotopy inverse to each other, we thus need natural transformations $\mu: Y*\Pi\longrightarrow \mathrm{Id}_{\mathsf{Rep}(\mathfrak{a})}$ and $\eta:\mathrm{Id}_{\mathsf{Rep}(\mathfrak{a})}\longrightarrow Y*\Pi$ that are inverse to each other. We inductively define the candidate cA_∞ natural transformations μ and η through the same formulas as the A_∞-natural transformations from [3, §8]. If we can show that, in our context, these are cA_∞-natural transformations, the rest of the proof (in particular, the fact that μ and η are inverse) will follow from the same arguments as in [3, §8]. We only give the proof for μ, because of the great similarity between μ and η. By construction, μ is a family

$$\begin{aligned}
&\mu_0: k\longrightarrow \mathsf{Rep}(\mathfrak{a})\Big(y(A),y(A)\Big)\\
&\mu_1: \mathsf{Rep}(\mathfrak{a})\Big(y(A),y(B)\Big)\longrightarrow \mathsf{Rep}(\mathfrak{a})\Big(y(A),y(B)\Big)\\
&\mu_2: \mathsf{Rep}(\mathfrak{a})^{\otimes 2}\Big(y(A),y(B)\Big)\longrightarrow \mathsf{Rep}(\mathfrak{a})\Big(y(A),y(B)\Big)\\
&\quad\vdots
\end{aligned}$$

defined as:
$\mu_0=\mathrm{Id}_{y(A)}$
$\mu_1(\xi)$ is the pre-natural transformation given by

$$\begin{aligned}
\big[\mu_1(\xi)\big]_n:\mathfrak{a}^{\otimes n}(D,C)&\longrightarrow \mathsf{PCom}(k)\Big(y(A)(C),y(B)(D)\Big)\\
x&\longmapsto\Big[\mathfrak{a}(C,A)\to\mathfrak{a}(D,B): g\mapsto\big(\xi_{n+1}(g,x)\big)(1_A)\Big]
\end{aligned}$$

$\mu_2(\xi,\nu)$ is the pre-natural transformation given by

$$\begin{aligned}
\big[\mu_2(\xi,\nu)\big]_n:\mathfrak{a}^{\otimes n}(D,C)&\longrightarrow \mathsf{PCom}(k)\Big(y(A)(C),y(B)(D)\Big)\\
x&\longmapsto\Big[\mathfrak{a}(C,A)\to\mathfrak{a}(D,B): g\mapsto\Big(\sum_a\xi_{1+n_{(2,a)}}\big[\big(\nu_{1+n_{(1,a)}}(g,x_a^1)\big)(1),x_a^2\big]\Big)(1)\Big]
\end{aligned}$$

where $n_{(i,a)} = |x^i_a|$. For all $k \geq 2$, $\mu_k(\xi^{(k)}, \dots, \xi^{(1)})$ is the pre-natural transformation given by

(26)

$$\begin{aligned}
\left[\mu_k(\xi^{(k)}, \dots, \xi^{(1)})\right]_n : \mathfrak{a}^{\otimes n}(D, C) &\longrightarrow \mathsf{PCom}(k)\Big(y(A)(C), y(B)(D)\Big) \\
x &\longmapsto \Big[\mathfrak{a}(C, A) \to \mathfrak{a}(D, B)\Big] \\
&\qquad g \mapsto \Big(\sum_a \xi^{(k)}_{1+n_{(k,a)}} \cdots \Big[\Big(\xi^{(2)}_{1+n_{(2,a)}}\big[\big(\xi^{(1)}_{1+n_{(1,a)}}(g, x^1_a)\big)(1), x^2_a\big]\Big)(1)\Big] \dots, x^k_a\Big)(1)
\end{aligned}$$

Analogously to the filteredness of the components of Π, we find that μ is a filtered object.

By the strict unitality of $\mathfrak{a}$, we have that

$$\begin{aligned}
0 &= \mu_1(\mathfrak{M}_0) = m(m_0, -)(g, *)(1) \\
&= \mu_2(-, \mathfrak{M}_0) = \mu_2(\mathfrak{M}_0, -) = \mu_3(-, -, \mathfrak{M}_0) = \dots
\end{aligned}$$

where $*$ is an element of $B\mathfrak{a}$. The rest of the proof is analogous to the proof in [3, §8] that μ is an A_∞-natural transformation. □

Lemma 4.16. *The composition* $\Pi * Y : \mathfrak{a} \longrightarrow \mathfrak{a}$ *equals the identity functor* $Id_{\mathfrak{a}}$*.*

Proof. We will prove that they have the same components $B_k\mathfrak{a} \longrightarrow \mathfrak{a}$, entailing their equality.

$$\begin{aligned}
(\Pi * Y)_0 &= 0 = (\mathrm{Id}_{\mathfrak{a}})_0 \\
(\Pi * Y)_1(f) &= \Pi_1(Y_1(f)) \\
&= \Pi_1(m(f, -)) \\
&= m_2(f, 1) \\
&= f \\
&= (\mathrm{Id}_{\mathfrak{a}})_1(f) \\
(\Pi * Y)_2(f, g) &= \Pi_1(Y_2(f, g)) + \Pi_2(Y_1(f), Y_1(g)) \\
&= \Pi_1(m(f, g, -)) + \Pi_2(m(f, -), m(g, -)) \\
&= m_3(f, g, 1) + m_3(f, 1, m_2(g, 1)) \\
&= 0 \\
&= (\mathrm{Id}_{\mathfrak{a}})_2(f, g)
\end{aligned}$$

Analogously, we find by means of the strict unitality of $\mathfrak{a}$ that the higher components match as well. □

4.3. **Curved tensor product of cA_∞-categories.** In this section, we define a tensor product between arbitrary cA_∞-categories and give an explicit alternative construction on the underlying tensor quiver in case one of the tensor factors is a cdg category.

We assume all cA_∞-categories in this section to be strictly unital.

Definition 4.17. Consider cA_∞-categories $\mathfrak{a}$ and $\mathfrak{b}$. We define the *Yoneda tensor product*

$$\mathfrak{a} \otimes_Y \mathfrak{b} = \mathsf{Rep}(\mathfrak{a}) \otimes_{cdg} \mathsf{Rep}(\mathfrak{b})$$

where the tensor product on the right hand side is the classical tensor product of cdg-categories.

Proposition 4.18. *Consider cA_∞-categories $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$, such that $\mathfrak{a} \cong \mathfrak{b}$ are cA_∞-homotopy equivalent. There is a homotopy-equivalence of cA_∞-categories*

$$\mathfrak{a} \otimes_Y \mathfrak{c} \cong \mathfrak{b} \otimes_Y \mathfrak{c}$$

Proof. By Theorem 4.15, we know that the cA_∞-homotopy equivalence $\mathfrak{a} \cong \mathfrak{b}$ results in a cA_∞-homotopy equivalence $\mathsf{Rep}(\mathfrak{a}) \cong \mathsf{Rep}(\mathfrak{b})$. Consider the cA_∞-functors $F : \mathsf{Rep}(\mathfrak{a}) \longrightarrow \mathsf{Rep}(\mathfrak{b})$, $G : \mathsf{Rep}(\mathfrak{b}) \longrightarrow \mathsf{Rep}(\mathfrak{a})$ and natural transformations μ, η and pre-natural transformation κ, κ' expressing the cA_∞-homotopy equivalence. I.e.

$$\mathfrak{M}_2(\mu, \eta) - \mathrm{Id}_{\mathrm{Id}} = m_1(\kappa)$$

$$\mathfrak{M}_2(\eta, \mu) - \mathrm{Id}_{FG} = m_1(\kappa')$$

Consider the functors $F \otimes \mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}$, $G \otimes \mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}$ (see Definition 4.21) and appropriate (pre-)natural transformations (see Proposition 4.24). By definition of the structure on $\mathsf{cFun}(\mathsf{Rep}(\mathfrak{a}) \otimes \mathsf{Rep}(\mathfrak{c}), \mathsf{Rep}(\mathfrak{b}) \otimes \mathsf{Rep}(\mathfrak{c}))$, we see that

$$\begin{aligned}\mathfrak{M}_2(\mu \otimes \mathrm{Id}_{\mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}}, \eta \otimes \mathrm{Id}_{\mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}}) - \mathrm{Id}_{\mathrm{Id}} \otimes \mathrm{Id}_{\mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}} &= \mathfrak{M}_2(\mu, \eta) \otimes \mathrm{Id}_{\mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}} - \mathrm{Id}_{\mathrm{Id}} \otimes \mathrm{Id}_{\mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}} \\ &= \mathfrak{M}_1(\kappa) \otimes \mathrm{Id}_{\mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}}\end{aligned}$$

and thus that $F \otimes \mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}$ and $G \otimes \mathrm{Id}_{\mathsf{Rep}(\mathfrak{c})}$ determine a cA_∞-homotopy equivalence $\mathfrak{a} \otimes_Y \mathfrak{c} \cong \mathfrak{b} \otimes_Y \mathfrak{c}$. ☐

In the remainder of this section, we present a direct construction of a tensor product, without reference to associated categories of representable modules, in case one of the tensor factors is a cdg category. Let $(\mathfrak{a}, m^{\mathfrak{a}})$ be a cdg-category, and $(\mathfrak{b}, m^{\mathfrak{b}})$ a cA_∞-category. We define a structure μ on $\mathfrak{a} \otimes \mathfrak{b}$ by

$$\begin{aligned}\mu_0 &= (m_0^{\mathfrak{a}}, 1_{\mathfrak{b}}) + (1_{\mathfrak{a}}, m_0^{\mathfrak{b}}) \\ \mu_1(a, b) &= (m_1^{\mathfrak{a}}(a), b) - (-1)^{|a|}(a, m_1^{\mathfrak{b}}(b)) \\ \mu_2((a, b), (a', b')) &= (m_2^{\mathfrak{a}}(a, a'), m_2^{\mathfrak{b}}(b, b')) \\ \mu_3((a, b), (a', b'), (a'', b'')) &= (m_2^{\mathfrak{a}}(a, m_2^{\mathfrak{a}}(a', a'')), m_3^{\mathfrak{b}}(b, b', b'')) \\ &\vdots\end{aligned}$$

Proposition 4.19. *The components μ_k define a cA_∞-structure μ on the tensor product $\mathfrak{a} \otimes \mathfrak{b}$.*

Proof. The components μ_k are filtered because the components $m_k^{\mathfrak{a}}$ and $m_k^{\mathfrak{b}}$ are filtered.

We have that μ satisfies the identity $\sum_{j+k+l=p} (-1)^{jk+l} \mu_{j+l+1}(1^{\otimes j} \otimes \mu_k \otimes 1^{\otimes l}) = 0$. By strict unitality, we have for $p = 0$ that

$$\begin{aligned}\mu_1(\mu_0) &= \mu_1((m_0^{\mathfrak{a}}, 1_{\mathfrak{b}}) + (1_{\mathfrak{a}}, m_0^{\mathfrak{b}})) \\ &= (m_1^{\mathfrak{a}}(m_0^{\mathfrak{a}}), 1_{\mathfrak{b}}) - (m_0^{\mathfrak{a}}, m_1^{\mathfrak{b}}(1_{\mathfrak{b}})) + (m_1^{\mathfrak{a}}(1_{\mathfrak{a}}), m_0^{\mathfrak{b}}) - (1_{\mathfrak{a}}, m_1^{\mathfrak{b}}(m_0^{\mathfrak{b}})) = 0\end{aligned}$$

for $p = 1$ that

$$\begin{aligned}
\mu_1(\mu_1)(a,b) &= \mu_1((m_1^{\mathfrak{a}}(a),b) - (-1)^{|a|}(a, m_1^{\mathfrak{b}}(b))) \\
&= (m_1^{\mathfrak{a}}(m_1^{\mathfrak{a}}(a)),b) - (-1)^{|a|}(m_1^{\mathfrak{a}}(a), m_1^{\mathfrak{b}}(b)) - (-1)^{|m_1^{\mathfrak{a}}(a)|}(m_1^{\mathfrak{a}}(a), m_1^{\mathfrak{b}}(b)) - (a, m_1^{\mathfrak{b}}(m_1^{\mathfrak{b}}(b))) \\
&= -(m_2^{\mathfrak{a}}(a, m_0^{\mathfrak{a}}) + m_2^{\mathfrak{a}}(m_0^{\mathfrak{a}}, a), b) - (a, m_2^{\mathfrak{b}}(b, m_0^{\mathfrak{b}}) + m_2^{\mathfrak{b}}(m_0^{\mathfrak{b}}, b)) \\
&= -\mu_2((a,b),\mu_0) + \mu_2(\mu_0,(a,b))
\end{aligned}$$

and for $p \geq 2$

$$\begin{aligned}
&\sum_{j+k+l=p} (-1)^{jk+l}\mu_{j+l+1}(1^{\otimes j} \otimes \mu_k \otimes 1^{\otimes l})((a_1,b_1),\dots,(a_p,b_p)) \\
&= \Big(m_2^{\mathfrak{a}}(a_1, m_2^{\mathfrak{a}}(\dots, m_2^{\mathfrak{a}}(a_{p-1},a_p))), \sum_{j+k+l=p} (-1)^{jk+l} m_{j+l+1}^{\mathfrak{b}}(1^{\otimes j} \otimes m_k^{\mathfrak{b}} \otimes 1^{\otimes l})(b_1,\dots,b_p)\Big) = 0
\end{aligned}$$

since the terms from $\mu_1\mu_k$ and $\mu_k\mu_1$ involving $m_1^{\mathfrak{a}}$ in the first component will cancel against each other since $\mathfrak{a}$ is a cdg-category. The terms involving $m_0^{\mathfrak{a}}$ will also cancel since these will occur as tensored with an element of the form $m_{p+1}^{\mathfrak{b}}(\dots, 1_{\mathfrak{b}}, \dots) = 0$. $\square$

We put $\mathfrak{a} \otimes_{c\infty} \mathfrak{b} = (\mathfrak{a} \otimes \mathfrak{b}, \mu)$ for the cA_∞-structure μ from Proposition 4.19. Clearly, if $\mathfrak{a}$ and $\mathfrak{b}$ are both l-curved for $l \in \mathbb{L}$, the same holds for $\mathfrak{a} \otimes_{c\infty} \mathfrak{b}$.

Remark 4.20. When $\mathfrak{a}$ is a cA_∞-category with non-zero higher components, the natural structure μ on the tensor product will produce mixed-terms in the expression $\sum_{j+k+l=p}(-1)^{jk+l}\mu(1^{\otimes j} \otimes \mu_k \otimes 1^{\otimes l})$ that will not cancel out against each other, whence it fails to define a cA_∞-structure.

Definition 4.21. Consider a unital cdg-functor $F : \mathfrak{a} \longrightarrow \mathfrak{c}$, and a unital cA_∞-functor $G : \mathfrak{b} \longrightarrow \mathfrak{d}$, with underlying morphisms f resp. g, and either F or G strict. The *tensor-functor* $F \otimes G : \mathfrak{a} \otimes \mathfrak{b} \longrightarrow \mathfrak{c} \otimes \mathfrak{d}$ is the qA_∞-functor with underlying morphism $f \otimes g$, defined by

$$(f \otimes g)(A,B) = (f(A), g(B))$$

and components on the morphism-sets, defined by

$$\begin{aligned}
(F \otimes G)_0 : k &\longrightarrow \mathfrak{c} \otimes \mathfrak{d} = F_0 \otimes 1_{\mathfrak{d}} + 1_{\mathfrak{c}} \otimes G_0 \\
(F \otimes G)_1 : \mathfrak{a} \otimes \mathfrak{b} &\longrightarrow \mathfrak{c} \otimes \mathfrak{d} = F_1 \otimes G_1 \\
(F \otimes G)_2 : (\mathfrak{a} \otimes \mathfrak{b})^{\otimes 2} &\longrightarrow \mathfrak{c} \otimes \mathfrak{d} = m_2(F_1, F_1) \otimes G_2 \\
(F \otimes G)_3 : (\mathfrak{a} \otimes \mathfrak{b})^{\otimes 3} &\longrightarrow \mathfrak{c} \otimes \mathfrak{d} = m_2(F_1, m_2(F_1, F_1)) \otimes G_3 \\
&\vdots
\end{aligned}$$

Proposition 4.22. *The tensor-functor $F \otimes G$ is a cA_∞-functor.*

Proof. The filteredness of the components $(F \otimes G)_k$ follows from the filteredness of the components F_k, G_k and m_2.

To prove that $F \otimes G$ defines a cA_∞-functor, we need to show that $\widehat{F \otimes G}$ is a dg-cocategory morphism. Since it clearly commutes with the comultiplication, we need to check whether $\hat{m}'\widehat{F \otimes G} = \widehat{F \otimes G}\hat{m}$.

$$
\begin{aligned}
\hat{m}'\widehat{F \otimes G}|_0 &= \mu'_0 + \mu'_1((F \otimes G)_0) + \mu'_2((F \otimes G)_0, (F \otimes G)_0) + \dots \\
&= (m_0^{\mathfrak{c}}, 1_{\mathfrak{d}}) + (1_{\mathfrak{c}}, m_0^{\mathfrak{d}}) + \big((m_1^{\mathfrak{c}}, -) + (-, m_1^{\mathfrak{d}})\big)\big(F_0 \otimes 1_{\mathfrak{d}} + 1_{\mathfrak{c}} \otimes G_0\big) \\
&\quad + (m_2^{\mathfrak{c}}, m_2^{\mathfrak{d}})\big((F_0 \otimes 1_{\mathfrak{d}} + 1_{\mathfrak{c}} \otimes G_0), (F_0 \otimes 1_{\mathfrak{d}} + 1_{\mathfrak{c}} \otimes G_0)\big) + \dots \\
&= (m_0^{\mathfrak{c}}, 1_{\mathfrak{d}}) + (1_{\mathfrak{c}}, m_0^{\mathfrak{d}}) + (m_1^{\mathfrak{c}}(F_0), 1_{\mathfrak{d}}) + (1_{\mathfrak{c}}, m_1^{\mathfrak{d}}(G_0)) + (m_2^{\mathfrak{c}}(F_0, F_0), 1_{\mathfrak{d}}) + (F_0, G_0) + (F_0, G_0) \\
&\quad + (1_{\mathfrak{c}}, m_2^{\mathfrak{d}}(G_0, G_0)) + (m_2^{\mathfrak{c}}(F_0, m_2^{\mathfrak{c}}(F_0, F_0)), m_3^{\mathfrak{d}}(1_{\mathfrak{d}}, 1_{\mathfrak{d}}, 1_{\mathfrak{d}})) + \dots \\
&\quad + (m_2^{\mathfrak{c}}(1_{\mathfrak{c}}, m_2^{\mathfrak{c}}(1_{\mathfrak{c}}, 1_{\mathfrak{c}})), m_3^{\mathfrak{d}}(G_0, G_0, G_0)) + \dots \\
&= (m_0^{\mathfrak{c}}, 1_{\mathfrak{d}}) + (1_{\mathfrak{c}}, m_0^{\mathfrak{d}}) + (m_1^{\mathfrak{c}}(F_0), 1_{\mathfrak{d}}) + (1_{\mathfrak{c}}, m_1^{\mathfrak{d}}(G_0)) + (m_2^{\mathfrak{c}}(F_0, F_0), 1_{\mathfrak{d}}) \\
&\quad + (1_{\mathfrak{c}}, m_2^{\mathfrak{d}}(G_0, G_0)) + (1_{\mathfrak{c}}, m_3^{\mathfrak{d}}(G_0, G_0, G_0)) + \dots \\
&= (F_1(m_0^{\mathfrak{a}}), G_1(1_{\mathfrak{b}})) + (F_1(1_{\mathfrak{a}}), G_1(m_0^{\mathfrak{b}})) \\
&= \widehat{F \otimes G}\hat{m}|_0
\end{aligned}
$$

$$
\begin{aligned}
\hat{m}'\widehat{F \otimes G}|_1 &= \mu'_1((F \otimes G)_1) + \mu'_2((F \otimes G)_0, (F \otimes G)_1) + \mu'_2((F \otimes G)_1, (F \otimes G)_0) \\
&\quad + \mu'_3((F \otimes G)_0, (F \otimes G)_0, (F \otimes G)_1) + \mu'_3((F \otimes G)_0, (F \otimes G)_1, (F \otimes G)_0) \\
&\quad + \mu'_3((F \otimes G)_1, (F \otimes G)_0, (F \otimes G)_0) + \dots \\
&= (m_1^{\mathfrak{c}}(F_1), G_1) + (F_1, m_1^{\mathfrak{d}}(G_1)) + \big(m_2^{\mathfrak{c}}(F_0, F_1) + m_2^{\mathfrak{c}}(F_1, F_0), G_1\big) + \big(F_1, m_2^{\mathfrak{d}}(G_0, G_1) + m_2^{\mathfrak{d}}(G_1, G_0)\big) \\
&\quad + \big(F_1, m_3^{\mathfrak{d}}(G_0, G_0, G_1) + m_3^{\mathfrak{d}}(G_0, G_1, G_0) + m_3^{\mathfrak{d}}(G_1, G_0, G_0)\big) + \dots
\end{aligned}
$$

$$
\begin{aligned}
\widehat{F \otimes G}\hat{m}|_1 &= (F \otimes G)_1\mu_1 + (F \otimes G)_2(-, \mu_0) - (F \otimes G)_2(\mu_0, -) \\
&= (F_1 \otimes G_1)\big((m_1^{\mathfrak{a}}, -) + (-, m_1^{\mathfrak{b}})\big) \\
&\quad + (m_2^{\mathfrak{c}}(F_1, F_1) \otimes G_2)\Big((-, (m_0^{\mathfrak{a}}, 1_{\mathfrak{b}}) + (1_{\mathfrak{a}}, m_0^{\mathfrak{b}})) - ((m_0^{\mathfrak{a}}, 1_{\mathfrak{b}}) + (1_{\mathfrak{a}}, m_0^{\mathfrak{b}}), -)\Big) \\
&= (F_1(m_1^{\mathfrak{a}}), G_1) + (F_1, G_1(m_1^{\mathfrak{b}})) + (m_2^{\mathfrak{c}}(F_1, F_1(m_0^{\mathfrak{a}})), G_2(-, 1_{\mathfrak{b}})) - (m_2^{\mathfrak{c}}(F_1(m_0^{\mathfrak{a}}), F_1), G_2(1_{\mathfrak{b}}, -)) \\
&\quad + (m_2^{\mathfrak{c}}(F_1, F_1(1_{\mathfrak{a}})), G_2(-, m_0^{\mathfrak{b}})) - (m_2^{\mathfrak{c}}(F_1(1_{\mathfrak{a}}), F_1), G_2(m_0^{\mathfrak{b}}, -)) \\
&= (F_1(m_1^{\mathfrak{a}}), G_1) + (F_1, G_1(m_1^{\mathfrak{b}})) + (F_1, G_2(-, m_0^{\mathfrak{b}})) - (F_1, G_2(m_0^{\mathfrak{b}}, -)) \\
&\ \ \vdots
\end{aligned}
$$

The equalities come from the fact that g is a unital cA_∞-functor, that f is a unital cdg-functor, and that one of them is strict.

□

Remark 4.23. In the proof, the signs coming from μ_1 have been supressed in the notation, but are respected, since F_1 is a morphism of degree 0.

Proposition 4.24. *Consider strict unital cdg-functors $F, G : \mathfrak{a} \longrightarrow \mathfrak{c}$ and unital cA_∞-functors $H, K : \mathfrak{b} \longrightarrow \mathfrak{d}$ with tensor-functors $F \otimes H$ and $G \otimes K$. Let $\eta : F \longrightarrow G$ and $\nu : H \longrightarrow K$ be a cdg- resp. a cA_∞-natural transformation. There exist a natural transformation $\eta \otimes \nu : F \otimes H \longrightarrow G \otimes K$.*

Proof. We define the natural transformation $\eta \otimes \nu$ by the collection

$$\begin{aligned}
(\eta \otimes \nu)_0 &= \eta_0 \otimes \nu_0 \\
(\eta \otimes \nu)_1 &= m_2(G_1, \eta_0) \otimes \nu_1 \\
(\eta \otimes \nu)_2 &= m_2(G_1, m_2(G_1, \eta_0)) \otimes \nu_2 \\
&\ \vdots
\end{aligned}$$

The filteredness of the components $(\eta \otimes \nu)_k$ follows from the filteredness of the components ν_k, η_0 and $m_2(G_1, -)$.

In order to show that $\eta \otimes \nu$ is a natural transformation, we have to check that $\mathfrak{M}_1(\eta \otimes \nu) = 0$. Using the unitality of the functors and natural transformations, we have

$$\begin{aligned}
\big(\mathfrak{M}_1(\eta \otimes \nu)\big)_0 &= \mu_1\big((\eta \otimes \nu)_0\big) + \mu_2\big((G \otimes K)_0, (\eta \otimes \nu)_0\big) + \mu_2\big((\eta \otimes \nu)_0, (F \otimes H)_0\big) \\
&\quad + \mu_3\big((G \otimes K)_0, (G \otimes K)_0, (\eta \otimes \nu)_0\big) + \mu_3\big((G \otimes K)_0, (\eta \otimes \nu)_0, (F \otimes H)_0\big) \\
&\quad + \mu_3\big((\eta \otimes \nu)_0, (F \otimes H)_0, (F \otimes H)_0\big) + \ldots - (\eta \otimes \nu)_1(\mu_0) \\
&= m_1(\eta_0) \otimes \nu_0 + \eta_0 \otimes m_1(\nu_0) + \eta_0 \otimes \big(m_2(K_0, \nu_0) + m_2(\nu_0, H_0)\big) + \big(m_2(G_0, \eta_0) + m_2(\eta_0, F_0)\big) \otimes \nu_0 \\
&\quad + m_2(G_0, m_2(G_0, \eta_0)) \otimes \underbrace{m_3(1_{\mathfrak{d}}, 1_{\mathfrak{d}}, \nu_0)}_{=0} + \ldots + m_2(1_{\mathfrak{b}}, m_2(1_{\mathfrak{b}}, \eta_0)) \otimes m_3(K_0, K_0, \nu_0) \\
&\quad + \ldots + m_2(1_{\mathfrak{b}}, m_2(1_{\mathfrak{b}}, \eta_0)) \otimes m_3(K_0, \nu_0, H_0) + \ldots + m_2(1_{\mathfrak{b}}, m_2(1_{\mathfrak{b}}, \eta_0)) \otimes m_3(\nu_0, H_0, H_0) \\
&\quad + \ldots - (m_2(G_1, \eta_0) \otimes \nu_1)(m_0^{\mathfrak{a}} \otimes 1_{\mathfrak{b}} + 1_{\mathfrak{a}} \otimes m_0^{\mathfrak{b}}) \\
&= \Big(m_1(\eta_0) + m_2(G_0, \eta_0) + m_2(\eta_0, F_0)\Big) \otimes \nu_0 \\
&\quad + \eta_0 \otimes \Big(m_1(\nu_0) + m_2(K_0, \nu_0) + m_2(\nu_0, H_0) + m_3(K_0, K_0, \nu_0) + m_3(K_0, \nu_0, H_0) \\
&\qquad\qquad + m_3(\nu_0, H_0, H_0) + \ldots - \nu_1(m_0^{\mathfrak{b}})\Big) \\
&= (\mathfrak{M}_1(\eta))_0 \otimes \nu_0 + \eta_0 \otimes (\mathfrak{M}_1(\nu))_0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\big(\mathfrak{M}_1(\eta \otimes \nu)\big)_1 &= \mu_1\big((\eta \otimes \nu)_1\big) + \mu_2\big((G \otimes K)_0, (\eta \otimes \nu)_1\big) + \mu_2\big((\eta \otimes \nu)_1, (F \otimes H)_0\big) + \mu_2\big((G \otimes K)_1, (\eta \otimes \nu)_0\big) \\
&\quad + \mu_2\big((\eta \otimes \nu)_0, (F \otimes H)_1\big) + \ldots - (\eta \otimes \nu)_1(\mu_1) - (\eta \otimes \nu)_2(-, \mu_0) + (\eta \otimes \nu)_2(\mu_0, -) \\
&= m_1(m_2(G_1, \eta_0)) \otimes \nu_1 + m_2(G_1, \eta_0) \otimes m_1(\nu_1) + m_2(G_1, \eta_0) \otimes m_2(K_0, \nu_1) + m_2(G_0, m_2(G_1, \eta_0)) \otimes \nu_1 \\
&\quad + m_2(G_1, \eta_0) \otimes m_2(\nu_1, H_0) + m_2(m_2(G_1, \eta_0), F_0) \otimes \nu_1 + m_2(G_1, \eta_0) \otimes m_2(K_1, \nu_0) \\
&\quad + m_2(\eta_0, F_1) \otimes m_2(\nu_0, H_1) + \ldots - m_2(G_1(m_1), \eta_0) \otimes \nu_1 - m_2(G_1, \eta_0) \otimes \nu_1(m_1) \\
&\quad - m_2(G_1, m_2(G_1(m_0^{\mathfrak{a}}), \eta_0)) \otimes \nu_2(-, 1_{\mathfrak{c}}) - m_2(G_1, m_2(G_1(1_{\mathfrak{a}}), \eta_0)) \otimes \nu_2(-, m_0^{\mathfrak{c}}) \\
&\quad + m_2(G_1(m_0^{\mathfrak{a}}), m_2(G_1, \eta_0)) \otimes \nu_2(1_{\mathfrak{c}}, -) + m_2(G_1(1_{\mathfrak{a}}), m_2(G_1, \eta_0)) \otimes \nu_2(m_0^{\mathfrak{c}}, -) \\
&= \Big(m_1(m_2(G_1, \eta_0)) + m_2(G_1(m_1), \eta_0)\Big) \otimes \nu_1 \\
&\quad + m_2(G_1, \eta_0) \otimes \Big(m_1(\nu_1) + m_2(K_0, \nu_1) + m_2(\nu_1, H_0) + m_2(K_1, \nu_0) + m_2(\nu_0, H_1) + \ldots - \nu_1(m_1) \\
&\qquad\qquad - \nu_2(-, m_0^{\mathfrak{c}}) + \nu_2(m_0^{\mathfrak{c}}, -)\Big) \\
&= \Big(m_2(m_1(G_1), \eta_0)) + m_2(G_1, m_1(\eta_0)) + m_2(G_1(m_1), \eta_0)\Big) \otimes \nu_1 + m_2(G_1, \eta_0) \otimes (\mathfrak{M}_1(\nu))_1 \\
&= \Big(m_2((\mathfrak{M}_0^G)_1, \eta_0)) + m_2(G_1, (\mathfrak{M}_1(\eta))_0)\Big) \otimes \nu_1 + m_2(G_1, \eta_0) \otimes (\mathfrak{M}_1(\nu))_1 \\
&= 0
\end{aligned}$$

Analogously, we find for all $n \geq 2$ that

$$\begin{aligned}
&\big(\mathfrak{M}_1(\eta \otimes \nu)\big)_n \\
&= \Big(m_2\big((\mathfrak{M}_0^G)_1, \dots, m_2(G_1, \eta_0)\big) + \dots + m_2\big(G_1, \dots, m_2((\mathfrak{M}_0^G)_1, \eta_0)\big) + m_2(G_1, \dots, m_2(G_1, (\mathfrak{M}_1(\eta))_0)\Big) \otimes \nu_1 \\
&\quad + m_2(G_1, \eta_0) \otimes (\mathfrak{M}_1(\nu))_n
\end{aligned}$$

□

Proposition 4.25. *Consider cdg categories $\mathfrak{a}$ and $\mathfrak{a}'$ and cA_∞-categories $\mathfrak{b}$ and $\mathfrak{b}'$, such that we have cA_∞-homotopy equivalences $\mathfrak{a} \cong \mathfrak{a}'$ and $\mathfrak{b} \cong \mathfrak{b}'$. We have cA_∞-homotopy equivalences $\mathfrak{a} \otimes_{c\infty} \mathfrak{b} \cong \mathfrak{a}' \otimes_{c\infty} \mathfrak{b}$ and $\mathfrak{a} \otimes_{c\infty} \mathfrak{b} \cong \mathfrak{a} \cong_{c\infty} \mathfrak{b}'$.*

Proof. This follows from Propositions 4.19, 4.22 and 4.24. □

Theorem 4.26. *For a cdg-category $\mathfrak{a}$ and a cA_∞-category $\mathfrak{b}$, we have a cA_∞-homotopy equivalence $\mathfrak{a} \otimes_{c\infty} \mathfrak{b} \cong \mathfrak{a} \otimes_Y \mathfrak{b}$.*

Proof. By Propositions 4.25 and 4.18, we have

$$\mathfrak{a} \otimes_{c\infty} \mathfrak{b} \cong \mathsf{Rep}(\mathfrak{a}) \otimes_{c\infty} \mathsf{Rep}(\mathfrak{b}) = \mathsf{Rep}(\mathfrak{a}) \otimes_{cdg} \mathsf{Rep}(\mathfrak{b}) = \mathfrak{a} \otimes_Y \mathfrak{b}.$$

□

5. Curved Bar- and Cobar-constructions

In this section, we define the bar and cobar constructions associated to a cA_∞-category $\mathfrak{a}$, which generalize the classical A_∞-notions as described in [6]. Like in the A_∞-case, these constructions can be used to obtain an adjunction, described in Corollary 5.8. In particular, for every cA_∞-category $\mathfrak{a}$ there is an associated cdg-category $\Omega B\mathfrak{a}$, the enveloping cdg category. These constructions have already been considered in the curved case, by Nicolas in [16] and by Positselski in [17], in more restrictive settings. The relation with the approach developed in [1] remains to be elucidated.

Throughout, all objects and morphisms are supposed to be filtered.

5.1. **Curved bar-construction.** Consider a complete augmented filtered dg-cocategory $\mathfrak{c}$ and a filtered cA_∞-category $\mathfrak{a}$, and associate with it the graded k-module $\mathrm{Hom}_k(\mathfrak{c}, \mathfrak{a})$, whose nth component is formed by the couples (f_0, f), where f is a k-linear filtered homogeneous morphism of degree n which annihilates the augmentation $\epsilon : k \longrightarrow \mathfrak{c}$, and f_0 a k-linear filtered morphism $k \longrightarrow \mathfrak{a}$ of degree n such that $\sum_{k\geq 0}(f_0)^{\otimes k} \in \hat{B}\mathfrak{a}$. To such a couple, we can associate a morphism f_+ of degree n, by

$$f_+(x) = \begin{cases} f_0(x) & \text{if } x \in \mathsf{Im}(\epsilon); \\ f(x) & \text{else.} \end{cases}$$

A straightforward calculation shows that we can endow this module with a cA_∞-structure given by

$$b_0 = (m_0^{\mathfrak{a}},\ 0)$$
$$b_1(f_0, f) = (m_1^{\mathfrak{a}} f_0,\ m_1^{\mathfrak{a}} f - (-1)^{|f|} f d_{\mathfrak{c}})$$
$$\text{for } n \geq 2: \ b_n\Big((f_{0,1}, f_1), ..., (f_{0,n}, f_n)\Big) = \Big(m_n^{\mathfrak{a}}(f_{0,1}, \dots, f_{0,n}),\ m_n^{\mathfrak{a}}\big((f_1)_+ \otimes \dots \otimes (f_n)_+\big)\Delta^{(n-1)}\Big)$$

We define *the set of twisting cochains* from $\mathfrak{c}$ to $\mathfrak{a}$, as the subset $\mathsf{Tw}(\mathfrak{c},\mathfrak{a}) \subset \mathrm{Hom}^1_k(\mathfrak{c},\mathfrak{a})$ of the homogeneous linear morphisms (f_0, f) of degree 1 such that they satisfy the Maurer-Cartan equation

$$\sum_{k\geq 0} b_k((f_0,f)^{\otimes k}) = 0. \tag{27}$$

Remark 5.1. We have to remark that, in general, this sum does not need to exist. Due to the completeness of $\mathfrak{a}$ we see that the existence of the sum (27) in $\mathfrak{a}$ imposes a condition on f_0.

Let cCodg be the curved category of complete augmented filtered dg-cocategories, with dg-cocategory morphisms. Such a morphism can equivalently be described as a $(F_0, F) : \mathfrak{c} \longrightarrow \mathfrak{d}$, where F is an augmented cocategory morphism (as defined in [11, 2.1.2]) annihilating the augmentation of $\mathfrak{c}$, and $F_0 : k \longrightarrow \mathfrak{d}$ of degree 1 such that the associated morphism F_+ is a morphism of cocategories and commutes with the differentials (i.e. $d_{\mathfrak{d}}F_+ = F_+d_{\mathfrak{c}}$). Let $\mathfrak{a}$ be a filtered cA_∞-category. Consider the functor

$$\mathsf{cCodg} \longrightarrow \mathsf{Set} : \mathfrak{c} \mapsto \mathsf{Tw}(\mathfrak{c},\mathfrak{a}) \tag{28}$$

and let $cA_\infty - \mathsf{Cat}$ be the category of cA_∞ categories and cA_∞-functors. From now on, we will simply denote the completed Bar-construction from Definition 2.32 by $B\mathfrak{a}$ instead of $\hat{B}\mathfrak{a}$.

Proposition 5.2. *The completed bar-construction defines a fully faithful functor*

$$B : cA_\infty\text{-}\mathsf{Cat} \longrightarrow \mathsf{cCodg} : \mathfrak{a} \mapsto B\mathfrak{a},$$

giving rise to isomorphisms $\mathsf{Tw}(-,\mathfrak{a}) \cong \mathsf{cCodg}(-, B\mathfrak{a})$*, natural in* $\mathfrak{a}$*. In particular, the functor* (28) *is representable with* $B\mathfrak{a}$ *as representative.*

Proof. Let $\mathfrak{c}$ be a complete augmented filtered dg-cocategory, and consider a morphism $(F_0, F) : \mathfrak{c} \longrightarrow B\mathfrak{a}$. By the universal property of the Tensor-category [11, Lemma 1.1.2.2], we know that F_0 and F are determined by the projections $f_0 = p_1F_0$ and $f = p_1F$, as is the associated morphism F_+. It is not hard to see that the projection of the associated morphism F_+ is in fact $(f_0, f)_+$. The condition that F_+ has to commute with the differentials can equivalently be described as $p_1F_+d_{\mathfrak{c}} = p_1d_{B\mathfrak{a}}F_+$, i.e. by the equations

$$m_0^{\mathfrak{a}} + \sum_{k\geq 1} m_k^{\mathfrak{a}}(f_+^{\otimes k}\Delta^{k-1}) = f_+d_{\mathfrak{c}}$$

where $m_0^{\mathfrak{a}}$ only counts when the argument is an element of $\mathsf{Im}(\epsilon)$. Since this is clearly equivalent to the expression

$$\sum_{k\geq 0} b_k((f_0,f)^{\otimes k}) = 0$$

we find the required isomorphism $\mathsf{Tw}(\mathfrak{c},\mathfrak{a}) \cong \mathsf{cCodg}(\mathfrak{c}, B\mathfrak{a})$ $\square$

Remark 5.3. By definition of the completed bar-construction we have an isomorphism

$$\mathsf{cFunc}(\mathfrak{a},\mathfrak{b}) \cong \mathsf{cCodg}(B\mathfrak{a}, B\mathfrak{b})$$

given by sending $(f_0, f_1, \ldots)$ to $(\hat{f}_0, \widehat{(f_1, \ldots)})$, where $\widehat{(f_1, \ldots)}$ denotes the map

$$B\mathfrak{a} \longrightarrow B\mathfrak{b} : x \mapsto \begin{cases} x & \text{if } x \in \mathsf{Im}(\epsilon) \\ \widehat{(f_0, f_1, \ldots)}(x) & \text{else} \end{cases}$$

and the effect of m_0 and f_0 should be interpreted as working on $k = \mathsf{Im}(\epsilon)$.

5.2. **Curved Cobar-construction.** Consider $\mathsf{cdg\text{-}Cat}$, the category of cdg-categories with cdg-functors. Restricting ourselves to such cdg-categories $\mathfrak{a}$, the functor (28) is also co-representable, namely by the cobar-construction $\Omega\mathfrak{c}$.

Definition 5.4. Let $\mathfrak{c}$ be an augmented dg-cocategory (see Definitions 2.38, 2.40). The *reduced cocategory* $\bar{\mathfrak{c}}$ is the cocategory $\mathfrak{c}/\mathsf{Im}(\epsilon)$, with the same objects as $\mathfrak{c}$, and Hom-sets given by the expression:

$$\bar{\mathfrak{c}}(A, B) = \mathfrak{c}(A, B)/\mathsf{Im}(\epsilon_{A,B}).$$

Remark 5.5. Let $A \neq B$ be objects of $\mathfrak{c}$. By definition of the augmentation, we have that $\mathsf{Im}(\epsilon_{A,B}) = 0$, and thus that $\bar{\mathfrak{c}}(A, B) = \mathfrak{c}(A, B)$.

Example 5.6. Consider an A_∞-category $\mathfrak{a}$, and its bar-construction $B\mathfrak{a}$ (as defined in [11]). The reduced bar-construction is given by

$$\overline{B\mathfrak{a}} = \bigoplus_{n \geq 1} (\Sigma\mathfrak{a})^{\otimes n}.$$

The augmentation $\epsilon : k \longrightarrow \mathfrak{c}$ yields a direct sum decomposition of $\mathfrak{c} = \mathsf{Im}(\epsilon) \oplus \bar{\mathfrak{c}}$. as such, we can decompose the comultiplication and differential on $\mathfrak{c}$ into the components

$$\Delta = \Delta_\epsilon + \Delta_{\bar{\epsilon}} \ ; \quad d = d_\epsilon + d_{\bar{\epsilon}}$$

We define the *cobar-construction* as the reduced tensor-category of the reduced cocategory $\bar{\mathfrak{c}}$

$$\Omega\mathfrak{c} = \overline{T}(\Sigma^{-1}\bar{\mathfrak{c}}) = \bigoplus_{k \geq 1} (\Sigma^{-1}\bar{\mathfrak{c}})^{\otimes k}.$$

The multiplication on $\Omega\mathfrak{c}$ is given by the concatenation, whereas the differential and curvature are determined by the differential and comultiplication on $\bar{\mathfrak{c}}$. Namely,

$$m_0^\Omega = \Big((s^{-1} \otimes s^{-1})\Delta_\epsilon - s^{-1} d_\epsilon\Big)\epsilon$$
$$m_1^\Omega = \Big((s^{-1} \otimes s^{-1})\Delta_{\bar{\epsilon}} s - s^{-1} d_{\bar{\epsilon}} s\Big)^\wedge$$

where the expressions Δ and d are appropriately composed with the projection $\mathfrak{c} \longrightarrow \bar{\mathfrak{c}}$, and $(-)^\wedge$ is the extension of a multiplication to the tensor-category. I.e.

$$(f)^\wedge(c_1, \ldots, c_n) = \sum_k (c_1, \ldots, f(c_k), \ldots, c_n)$$

This is indeed a cdg-structure since d is a differential, and the comultiplication is coassociative.

Proposition 5.7. *The cobar-construction defines a fully faithful functor*

$$\Omega : \mathit{cCodg} \longrightarrow \mathit{cdg\text{-}Cat} : \mathfrak{c} \mapsto \Omega\mathfrak{c},$$

giving rise to isomorphisms $\mathsf{Tw}(\mathfrak{c}, -) \cong \mathit{cdg\text{-}Cat}(\Omega\mathfrak{c}, -)$*, natural in* $\mathfrak{c}$*. In particular, the functor* (28) *is co-representable with corepresentative* $\Omega\mathfrak{c}$*.*

Proof. Consider a morphism $f : \mathfrak{c} \longrightarrow \mathfrak{d}$. The associated morphism $\Omega f : \Omega\mathfrak{c} \longrightarrow \Omega\mathfrak{d}$ is defined $\Omega f(c_1, \dots, c_n) = (f_+(c_1), \dots, f_+(c_n))$. Since f is a morphism in cCodg, it is by the definition of the differential clear that Ωf is a cdg-functor

Let $(F_0, F) : \Omega\mathfrak{c} \longrightarrow \mathfrak{a}$ be a map in $\mathsf{cdg\text{-}Cat}(\Omega\mathfrak{c}, \mathfrak{a})$. Expressing that this is a cdg-functor, yields the classical identities

$$\begin{aligned} m_0^{\mathfrak{a}} + m_1^{\mathfrak{a}}(F_0) + m_2^{\mathfrak{a}}(F_0, F_0) &= F(m_0^{\Omega}) \\ m_1^{\mathfrak{a}}(F) + m_2^{\mathfrak{a}}(F, F_0) + m_2^{\mathfrak{a}}(F_0, F) &= F(m_1^{\Omega}) \\ m_2^{\mathfrak{a}}(F, F) &= F(m_2^{\Omega}) \end{aligned} \tag{29}$$

By definition of the multiplication on $\Omega\mathfrak{c}$, one finds that the component of F working on $\bar{\mathfrak{c}}^{\otimes k}$, $F_k : \bar{\mathfrak{c}}^{\otimes k} \longrightarrow \mathfrak{a}$, is given by an iteration of the third identity (29), and thus that the morphism $F : \Omega\mathfrak{c} \longrightarrow \mathfrak{a}$ is completely determined by it's first component $F_1 = F\iota$ with $\iota : \bar{\mathfrak{c}} \longrightarrow \Omega\mathfrak{c}$ the embedding. Since $\mathfrak{a}$ is a cdg-category, one sees that (F_0, F) is a cdg-functor if and only if

$$\sum_k b_k(F_0, F_1) = 0$$

As such we obtain an isomorphism

$$\mathsf{cdg\text{-}Cat}(\Omega\mathfrak{c}, \mathfrak{a}) \longrightarrow \mathsf{Tw}(\mathfrak{c}, \mathfrak{a}) : (F_0, F) \mapsto (F_0, F\iota)$$

The inverse to this map is given by

$$\mathsf{Tw}(\mathfrak{c}, \mathfrak{a}) \longrightarrow \mathsf{cdg\text{-}Cat}(\Omega\mathfrak{c}, \mathfrak{a})(f_0, f) \mapsto (f_0, g)$$

where g is the extension of f to $\Omega\mathfrak{c}$ by means of the expression (29). □

Corollary 5.8. *The bar and cobar constructions*

$$\begin{tikzcd} \mathit{cdg\text{-}Cat} \arrow[r, bend left, "B"] & \mathit{cCodg} \arrow[l, bend left, "\Omega"] \end{tikzcd}$$

form a pair of adjoint functors.

5.3. **The enveloping cdg-category.** Consider a cA_∞-category $\mathfrak{a}$. By the adjunction of Corollary 5.8, we have an isomorphism

$$\mathsf{cdg\text{-}Cat}(\Omega B\mathfrak{a}, \Omega B\mathfrak{a}) \cong \mathsf{cCodg}(B\mathfrak{a}, B\Omega B\mathfrak{a}) \cong \mathsf{cFun}_\infty(\mathfrak{a}, \Omega B\mathfrak{a})$$

which sends $(0, F) \in \mathsf{cdg\text{-}Cat}(\Omega B\mathfrak{a}, \Omega B\mathfrak{a})$ to the strict cA_∞-functor described by the morphism $f = F\iota : B\mathfrak{a} \longrightarrow \Omega B\mathfrak{a}$. As such the identity morphism $\mathrm{Id} : \Omega B\mathfrak{a} \longrightarrow \Omega B\mathfrak{a}$ corresponds to a cA_∞-functor $I : \mathfrak{a} \longrightarrow \Omega B\mathfrak{a}$ described by

$$\forall n \geq 1 : \ I_n : \mathfrak{a}^{\otimes n}(A, B) \longrightarrow \Omega B\mathfrak{a}(A, B) : (f_1, \dots, f_n) \mapsto \Big((f_1, \dots, f_n), 0, \dots\Big)$$

Remark 5.9. By definition of this morphism, it is universal among the cA_∞-morphisms from $\mathfrak{a}$ into a cdg-category.

Remark 5.10. By definition of the filtration on $\Omega B\mathfrak{a}$, it is clear that the components of I are filtered.

The adjunction of Corollary 5.8 also gives an isomorphism

$$\mathsf{cCodg}(B\mathfrak{a}, B(\mathsf{Rep}(\mathfrak{a}))) \cong \mathsf{cdg\text{-}Cat}(\Omega B\mathfrak{a}, \mathsf{Rep}(\mathfrak{a}))$$

which sends the Yoneda-functor $Y : \mathfrak{a} \longrightarrow \mathsf{Rep}(\mathfrak{a})$, defined in Proposition 4.13, to a strict cdg-functor $\Omega B\mathfrak{a} \longrightarrow \mathsf{Rep}(\mathfrak{a})$. If we compose this functor with the Yoneda-projection, defined in Theorem 4.15, we obtain the strict cA_∞-functor

$$P : \Omega B\mathfrak{a} \longrightarrow \mathfrak{a}$$

which has the identity as underlying morphism.

By means of these functors, we can now formulate the next theorem.

Theorem 5.11. *The strict functors*

$$I : \mathfrak{a} \longrightarrow \Omega B\mathfrak{a}$$

$$P : \Omega B\mathfrak{a} \longrightarrow \mathfrak{a}$$

are cA_∞-homotopy inverse to each other

Proof. Since both functors I and P can be factorized over the category $\mathsf{Rep}(\mathfrak{a})$ by means of the Yoneda functor Y and Yoneda projection Π, we can, in analogy with the proof of the curved Yoneda cA_∞-homotopy equivalence in Theorem 4.15, use the descriptions of the A_∞-natural transformations to obtain the desired cA_∞-natural transformations expressing the cA_∞-homotopy equivalence. $\square$

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